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ABSTRACT

The first of six chapters in this yearbook discusses articulation between junior and senior high school in terms of several specific content and topic areas. The following chapter is a selected review of research in mathematics education for the years 1915-1931. Investigations are categorized into one of nine areas and a 132-entry bibliography is included. The third chapter is devoted to a wide-ranging discussion of intuitive geometry. Chapter 4 suggests curriculum considerations for grades 7-12 with differentiation for varying ability levels. The next chapter presents the mathematics studied in German schools along with the methods, apparatus, and models used. Organization of a unit of instruction is the topic of the last chapter, and a detailed example is given using direct linear variation as the subject. (LS)

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

THE EIGHTH YEARBOOK

THE TEACHING OF MATHEMATICS IN THE SECONDARY SCHOOL

BUREAU OF PUBLICATIONS
Teachers College, Columbia University
NEW YORK CITY
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EDITOR'S PREFACE

This is the eighth of a series of Yearbooks which the National Council of Teachers of Mathematics began to publish in 1926. The topics covered in the preceding volumes are as follows:

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The purpose of the Eighth Yearbook is to present some of the important phases of the teaching of mathematics in the secondary school, particularly those that have not been treated in current magazines as fully as they are discussed here. Space does not permit us to treat as many different topics as we should like to present, but it is hoped that what is included will be of assistance to many persons interested in the secondary school.

I wish to express my personal appreciation as well as that of the National Council to all contributors who have helped to make this Yearbook possible.

W. D. REEVE

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THE TEACHING OF MATHEMATICS IN THE SECONDARY SCHOOL

ARTICULATION OF JUNIOR AND SENIOR HIGH SCHOOL MATHEMATICS

By E. R. BRESLICH
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The need of reorganizing traditional mathematics. Algebra, geometry, and trigonometry as taught in our high schools were originally organized for students in colleges. Later they were taken over by the college preparatory schools and high schools, being offered at first in the senior and junior years. Gradually they were moved downward to the early years of the secondary schools. They were presented to high school students almost as they had been taught in the colleges. When it was found that high school pupils encountered serious difficulties in the study of algebra and geometry, efforts were made to reorganize or to reconstruct these courses.

Because our school system was organized on the 8–4 plan, it seemed to be impossible to find a solution of the problem. For, on account of the gap existing between the elementary and high schools, the teaching of algebra and geometry had to be deferred to the ninth and tenth school years. This made it necessary to crowd a large amount of algebra into a year's time and to force the pupils to advance at a rate too rapid for understanding and assimilation. Since all the time of the second year was needed to cover the extensive content of demonstrative geometry, algebra had to be completely dropped during that period.

Simplification of the problem of reorganization by the junior high school. Attempts have been made by individual schools to solve the problem by offering some algebra in the last year of the elementary school. However, such efforts usually failed for two reasons: high school algebra was not easily adapted to the interests and abilities of elementary school pupils; and high school teachers were unwilling to accept algebra taught in the eighth grade as an equivalent to high school algebra. Often it was completely disregarded and the work done in the eighth grade was repeated in the ninth grade.

The development of the junior high school offered a real opportunity for the reorganization of secondary school mathematics, making it possible to extend downward the period of secondary education, to give instruction in algebra and geometry at an early period, and to distribute the content of these courses over a period of several years. This involved a reconstruction of the mathematical curriculum of the secondary school, which would offer introductory work in algebra and geometry in Grades 7 and 8 and the more advanced work in the later grades.

The gap between junior and senior high school mathematics. The reorganization necessitated the training of teachers qualified to teach the new courses; elementary school teachers were unfamiliar with the mathematics of the secondary school and high school teachers were not accustomed to pupils of elementary school age. To satisfy this need, courses in junior high school mathematics were offered by the universities. A new type of mathematics was being developed for the junior high school, while the senior high school continued their program of traditional courses in plane geometry, advanced algebra, solid geometry, and trigonometry.

The administration and supervision of the mathematics in the junior and senior high schools is usually not carried on by the same supervisor, and the teaching is done by two distinct faculties. Neither division knows much about the other.

The development of two types of secondary school mathematics, the separation of the junior high school from the senior high school, and the creation of two distinct faculties teaching the courses tend to cause a gap in secondary school mathematics fully as serious as that which formerly existed between the mathematics of the elementary and high schools. Steps should be taken to bring about a closer coöperation between these faculties.

Uniformity of administration as an essential factor in securing improvement. An ideal situation is to have but one department and one supervisor to direct it. When there are two supervisors, they should feel compelled by professional interest to develop a program to unify and articulate the work of the two departments. Mutual understanding and harmonious coöperation of the teachers will not be difficult to secure if there is uniformity in administration. To attain it will require frequent joint departmental conferences in which the problems of concern to both divisions are

freely discussed. The junior high school teacher will thus become acquainted with the content and objectives of the higher courses and will be able to do his work more intelligently. The senior high school teacher will acquire an understanding of the problems and limitations of pupils of the lower grades and will become willing to assume a just share of responsibility toward them. Both will grow in knowledge of subject matter. They will reach agreement on methods of teaching so that a pupil passing from one institution to the next higher need experience no sudden and radical change in teaching and study.

Necessity of continuity of instruction. Continuity in secondary school mathematics is exceedingly important. Teachers should therefore cease to regard junior and senior mathematics as distinct types. The two should be thoroughly merged. One course of study should be planned to include both, and the details should be worked out by joint committees selected from both faculties. As the pupil passes from the junior to the senior high school, the change in mathematics should be no greater than that usually experienced when he advances a grade. The senior high school course should begin exactly where that of the junior high school ends. The pupil's attitude toward mathematics in the senior high school will be friendly, for he has had enough mathematics to know what it is like and, since he has deliberately chosen to continue the study, he must be interested in it, feel capable to do the work, and believe that he will be benefited.

Articulation of junior and senior high school geometry. High school teachers of mathematics are sometimes heard to complain that it is difficult to teach geometry to pupils coming from the junior high school. They assert that the most interesting part of geometry has been taught in the lower school and that the attitude of the pupil is, therefore, one of indifference and exceedingly hard to change. Usually the difficulty arises from a lack of coöperation of the two departments, and the solution lies in establishing better understanding between them.

It is generally agreed that the course in demonstrative geometry has too much content to attain the major objectives with a large number of pupils. Many pupils capable of doing high school work never develop the ability to solve original exercises and have to be satisfied with learning and repeating the finished proofs found in the textbooks. It seems that teachers of high school geometry

should, therefore, welcome the opportunity of having some of the geometric work done in the lower grades.

From the point of view of the junior high school, the development of a course in geometry for the lower grades is not only desirable but imperative, because pupils should be taught to understand and interpret the geometric situations which they meet in the activities of everyday life and in various school subjects. Familiarity with the most common geometric figures, knowledge of an essential geometric vocabulary, appreciation of the beauty of geometric forms in nature, training in space imagination, ability to measure and to estimate size, acquaintance with a number of important geometric facts and relationships, and skill in the use of the geometric measuring instruments are all included among the important social and vocational objectives of the junior high school. These phases of geometry are needed in the activities of pupils and are also an excellent preparation for, and introduction to, high school geometry. Pedagogically, the junior high school period is the best in which to teach them.

The teaching of geometry is not always taken as seriously by junior high school teachers as the more traditional work in arithmetic. The content of the junior high school course is a matter that must not be neglected; it should be so definite that the senior high school teacher may know exactly what has been taught and what he may take for granted in his courses. In addition, the subject matter should be subjected to a careful program of testing and reteaching to secure objective evidence of what has been learned. Where this is done, the study of geometry will proceed without interruption when the pupil passes from the junior high school to the senior high school. Several examples will illustrate further the need of articulation between junior and senior high school geometry.

The fundamental concepts of geometry. A failure to understand the meaning of geometric concepts is the source of much difficulty in the subject in high school. A definition of the new terms and a few illustrations are often the only explanation that is given to students at this time. Furthermore, a considerable number of unfamiliar terms have to be crowded into the beginning of the high school course. To the pupil this type of work is uninteresting and often distasteful. Failing to catch the meanings, he memorizes statements which he does not understand. He is unable to make

correct use of them when they occur later in the logical discussions. He becomes discouraged and fails.

When the meanings of concepts are acquired in the junior high school, this situation may be entirely changed. There new terms arise in concrete situations and activities. They do not crowd each other. Ample time is allowed to observe, to compare, to measure, to draw, and to analyze. Understanding and satisfaction are the results. The list of concepts whose meanings have been established should be available to the high school teacher for inspection. He may then add other terms gradually as need for them arises in his course.

Constructional geometry. Traditionally, geometric constructions form an essential part of high school geometry. The present tendency is to introduce them early in the course rather than to defer them until they may be proved by logical demonstration. It is, therefore, a simple matter to move at least the so-called fundamental constructions downward into the junior high school. They give the pupil excellent practice with geometric instruments, and are helpful in making diagrams and designs. They may be used with sufficient frequency to be learned and retained. The simple repetition of these constructions will not be necessary in high school geometry, but they will be used in making constructions of a more complicated type.

Simple geometric facts. The pupil has become acquainted with numerous geometric facts and theorems through his daily life experiences before he enters the high school. Others he has learned in mensurational geometry in the elementary school. As far as he knows, the truth of these facts has never been questioned and he has made much use of them in the solution of problems. At the beginning of demonstrative geometry he is practically told that he must forget all the geometry he has previously acquired and that he will not be permitted to use any fact that has not been established by logical proof. This seems to him most illogical and confusing. Often he becomes openly antagonistic. He is in no frame of mind to appreciate or enjoy the beauty of the logical system of geometry.

The difficulty is easily removed by a little coöperation between the junior and senior high schools. An agreement should be reached on the particular facts to be taught in the junior high school. These facts should then be included among the assumptions with

which the course in demonstrative geometry begins in the senior high school. This increases the number of assumptions. Moreover, it enables the teacher to organize a course which observes a truly logical sequence. It may call for slight readjustments of the theorems in the textbook. Some of them will be reduced from the ranks of basic theorems and included among the exercises. No harm will be done. This change will conform to the tendency that has grown up in recent years of diminishing the number of basic theorems.

Developing the meaning of a logical demonstration. To a large extent demonstrative geometry is really a course in logic. The pupil entering the tenth grade has developed reasoning ability; he can hold his own in an argument and in a debate. However, the formal demonstration so prominent in tenth grade geometry is new to him. He comes upon it abruptly, without having had any former experiences that are similar. Many pupils find the step from junior high school geometry to logical geometry very difficult and go about their work aimlessly for weeks before they grasp the idea of the usefulness of logical demonstration. Often the inability to take this step is the cause of failure.

The problem is serious enough to call for careful consideration by the teachers of both schools. Somewhere in the course provision should be made for a period of instruction in which space and logic are being gradually joined. Modern textbooks on geometry take recognition of this need by offering an introductory chapter preceding the formal geometry. However, the results are not at all satisfactory because it takes more time to develop the meaning of logical demonstration than can be devoted to it in an introductory chapter.

The chances to solve the problem in the junior high school are much better. There the progress in the study of geometry is slow. Ever so many situations present themselves in which facts may be easily established by simple reasoning cycles. This work can be made attractive to the pupils. Gradually they will see and appreciate the power and advantage to be derived from the logical proof. It is possible to work out a detailed, definite program which will lead the pupil gradually from the method of direct observation through a period of informal reasoning to the stage of demonstrative geometry. The last belongs properly to the senior high school. The first two should be provided in the junior high school.

Continuity of instruction in arithmetic during the secondary school period. The fundamental processes of arithmetic are taught in the first six grades of the elementary school. However, arithmetical maturity is usually not attained by pupils before the junior or senior years of the high school. It is not unusual to find high school seniors who have not advanced beyond the level of the fifth grade pupil in arithmetic, although sometimes pupils are found in the seventh grade who have already attained maturity in arithmetic. Evidently, instruction in arithmetic must continue in secondary school mathematics even though a formal course may not, or should not, be offered. Anyone who has examined the results of arithmetic tests administered to high school pupils will agree to this. High school teachers cannot wave aside the responsibility. Arithmetical deficiencies of individuals and of classes should be discovered by means of tests given each year to each grade. Steps should then be taken for remedial work. Every teacher from the seventh grade up should know definitely his obligations in respect to arithmetic, and the faculties of both schools should coöperate to the fullest extent in the program of bringing about arithmetical improvement.

The laying of the foundation of senior high school mathematics in the junior high school. The writer does not intend to give the impression that the major aim of junior high school mathematics is preparation for the later courses. The content of the curriculum at any stage should be determined first of all by the needs of pupils in their activities in and out of school. However, this does not imply that the needs of later life and of the more advanced mathematics should be disregarded. Frequently the immediate and deferred needs are identical. Even if they are not, the latter demand and deserve careful consideration. They raise problems of teaching that are of importance to the pupil's successful continuation of the study of mathematics. If the faculties of both schools are interested and coöperate in the solution of these problems, a correct start will be made and instruction will continue without interruption and with a minimum loss of time and effort.

It is evident that teachers are not qualified to teach junior high school mathematics unless they have a thorough knowledge of the mathematics taught in the senior high school and in the junior college. Without this knowledge they will lack a comprehensive point of view, and their work will be ineffective. A few examples will illustrate the correctness of this assertion.

Continuity of the teaching of graphic representation. The teaching of graphs in secondary school mathematics is a relatively recent innovation. Originally they belonged to the field of higher mathematics, but they found the way into high school mathematics and from there into the junior high school. To-day they are taught, or used, in all mathematics courses and in some of the other school subjects. Their importance is recognized not only in school work but also in everyday life. The teaching of graphs begins in the junior high school and continues in the senior high school. The responsibility for teaching them is assumed by both departments, each taking its share.

In the junior high school the pupil should acquire a knowledge of different types of graphs used to represent numerical facts. He should attain considerable ability to employ them in a variety of situations and to understand and interpret relationships when they are pictured graphically. He should become familiar with the bar graph, line graph, and circular graph, and learn to solve linear and quadratic equations by graphic methods. This cannot be accomplished by the study of an occasional chapter on graphs. They should be used constantly. On the part of the teachers, it presupposes a thorough knowledge of the subject.

Graphic work begins when pupils draw line segments of given lengths and find lengths of given line segments by estimating and by applying the ruler. From this simple drawing and measuring of single line segments, but a small step remains to the statistical graph. This relation is not always recognized by the teachers, and the introductory work is often slighted and even omitted because its importance is not realized. The place for teaching it is in the early part of the seventh grade.

The drawing and measuring of line segments should be assigned a prominent place in mathematics, not only because it is an introduction to graphic work, but on account of its usefulness for other purposes. For example, it is essential to an understanding of such concepts as ratio of geometric magnitudes, proportion, similarity, and trigonometric function. If in trigonometry the sine function is introduced as a ratio of two sides of a right triangle, it is assumed that the pupil knows what "ratio of two sides" means. This seems justifiable because the term is used over and over in tenth grade geometry. The chances are that the teacher of geometry also assumed an understanding which did not exist, and failed to explain

the term. Most likely he disposed of it by saying that the ratio of two line segments is the ratio of the numerical measures. However, the explanation means little or nothing to the pupil who has not had actual experiences with measuring and with finding ratios by dividing one measure by another. Thus, the drawing and measuring of line segments is an introduction not only to graphic representation but also to the subjects of geometry, trigonometry, and analytics.

A program for teaching graphs during the entire period of secondary school mathematics should be worked out to make them a powerful tool in the study of mathematics and a mode of thinking which makes abstract relationships concrete. Not very long ago teachers objected to graphic representation on the ground that it introduced rather than removed difficulties. In textbooks on algebra the graphic solutions of equations were cautiously placed in a chapter by themselves where they could easily be omitted. In trigonometry the graphs of the trigonometric functions were regarded as frills. They were taken up when the course was finished, if time permitted.

The tendency to-day is to make graphs useful and to teach them not as curiosities but as an essential part of the course. The trigonometric functions illustrate this strikingly. To give the learner a clear conception of the functions and their relationships to each other, teachers make use of a variety of devices which may all be replaced by a simple graph. The student who understands graphs sees at a glance that the sine function is positive in the first two quadrants and negative in the last two; that it assumes all values from 0 to 1 as the angle changes from 0° to 90°; that the maximum and minimum values are $+1$ and -1; that the sine of $(n \times 90^\circ \pm x)$ is numerically equal to sine x or cos x according as n is even or odd; and that it is a periodic function. Moreover, the graph aids the mind to retain all these facts and, if in doubt, the student can reproduce them in a moment by making a rough picture of the graph.

The importance of graphic representation cannot be overlooked, and the coöperation of senior and junior high school teachers is needed if it is to be taught effectively.

The united efforts of junior and senior high school teachers needed to teach the formula. The formula deserves a prominent place in all mathematics courses. Instruction in it should start

early and never cease, for some time is required to develop the various abilities with formulas.

One of these is the ability to derive formulas. In senior high school mathematics formulas are rarely used unless they have been derived by the pupils. The same is true for the formulas which occur in the science courses. Experience shows that the deriving of the formula aids the pupil in understanding it and gives him confidence in its use. For this reason the pupils generally develop the progression formulas, the quadratic equation formula, the binomial theorem, and the laws of uniform motion and of falling bodies.

The first formulas with which pupils become acquainted are simple and easily derived. This is true particularly of the formulas for finding areas of the common plane figures and volumes of solids and for computing percentage and interest. Nevertheless, teachers and textbooks often neglect to derive them. They merely state the formulas, explain the meaning of the symbols, and proceed to the applications. Thus the pupil's conception of the formula is necessarily limited, since he sees in it little more than a rule or device which he must learn to manipulate to find the required answers to given problems. The opportunity for training in deriving formulas has been lost.

The evaluation of formulas is another process to be stressed throughout the entire period of secondary school mathematics. It deepens the pupil's understanding of mathematical symbols and relationships contained in formulas and gives practice in substitution and in the fundamental operations. This phase of formula work has always received its share of attention from the teachers.

The transformation of formulas is a third ability to be developed. Training may begin as soon as the pupil has derived the first formulas. Thus, he should be taught early to solve $c = 2\pi r$ for r; $i = prt$ for p, r, or t; $d = rt$ for r or t; $A = bh$ for b or h. It will take a large variety of formulas and problems, as well as constant practice and teaching extending over a period of years, to develop proficiency in transforming formulas.

A fourth phase of the study of formulas is the development of power to understand relationships contained in them. This is essential to mastery of the proper concepts. Thus, the pupil should see that the value of c is doubled in $c = 2\pi r$ if r is doubled, but that A in the formula $A = \pi r^2$ becomes 4 times as great if r is doubled;

that t in $t = \frac{d}{r}$ is increased if r is decreased, but that it is decreased if r is increased.

Relationships in formulas have received little attention in teaching, with a few exceptions, as in the case of the quadratic formula. Here it is shown that the character of the root of an equation is related to, and depends on, the values of the coefficients. Although the relationship is simple, pupils usually experience considerable difficulty in comprehending it. The reason is that they have not been trained to examine formulas in respect to the relationships existing among the variables. The preceding courses should provide such training.

The solution of verbal problems as a goal of instruction at all times. Proficiency in solving problems is regarded by many teachers as the most important function of the teaching of mathematics. They would make the problem the outstanding feature of each of the various courses. These teachers would introduce new processes with problems in which they occur and justify the processes on the ground that they are needed in solving the problems. Others emphasize verbal problems on account of their informational value. They regard them as the most useful and most interesting part of mathematics.

On the other hand, the teaching of verbal problems has been severely criticized for two reasons: the failure to select more problem material that will be understood and appreciated by the pupils, and the lack of an effective technique of teaching pupils to solve problems. Both criticisms necessitate the careful attention and coöperation of the teachers, for training in problem solving must continue without interruption in all mathematics courses. Problems should be more than mathematical puzzles. They should give the pupil a feeling of reality and impress him with the value and importance of mathematics.

Equally important is an effective technique of training pupils to solve problems. The literature relating to difficulties experienced by pupils in solving verbal problems is full of helpful suggestions. In the beginning of the secondary school period problems are solved by arithmetic. Later, the algebraic method is used almost exclusively. Somewhere within this period the transition must be made from the first method to the second.

However, the two methods have certain steps in common, and

practice in them should be provided throughout the entire period. This includes training in reading verbal problems understandingly, in comprehending the social situations contained in them, in analyzing the content of problems for the facts that are stated or implied, and in separating the known facts from those that are to be found. In addition to this, the arithmetical method requires training in selecting the right process and the algebraic method calls for practice in recognizing relationships and in deriving the equation, or equations, for solving each problem. The change from the first to the second method should be gradual. It is unwise to force upon the pupil a method which he does not appreciate. If he is led to see the superiority of the algebraic over the arithmetical method, he will prefer to use it.

The first step toward the application of the algebraic method will be the use of literal numbers to denote the unknown or required numbers. Somewhat later all unknown numbers will be expressed in terms of the same literal number to simplify the verbal statements. As a further simplification, the equation should be introduced. For the first problems the equations will be simple enough to be solved by inspection without the use of axioms or laws. As the problems increase in complexity, axioms will be employed in solving them. Progress should be gradual and sufficient time for assimilation should be allowed each step. At the end of the junior high school period the pupil should be quite proficient in solving verbal problems by algebraic methods.

The stressing of functional thinking throughout the period of secondary mathematics. In the past much more attention has been given to the computative and formal phases of mathematics than to the informational, cultural, and functional aspects. Without disregarding the importance of the first two phases, the tendency in modern teaching is to give more recognition to the last three. Thus the student of mathematics should acquire power to think independently, to learn to enjoy the beauty of mathematics, and to appreciate the subject as one of the great achievements of man. One of the objectives of the teaching of mathematics is the power to do functional thinking.

Until recently it was assumed that the function concept belongs to the field of higher mathematics. It is now generally agreed that training in functional thinking deserves a place in secondary school mathematics. Indeed, anyone who watches the activities of young

children will find evidence of functional thinking in the earliest stages of school work and in the everyday experiences of the pupils. These experiences should be utilized to give training in functional thinking. In the junior high school, opportunities for further training are numerous, especially in the study of algebraic formulas and the geometric theorems which express relationships between variables. At the senior high school level, the function concept may then be made the central theme of mathematics. "The teacher should have this idea constantly in mind and the pupils' advancement should be consciously directed along the lines which will present first one and then another of the ideas upon which finally the formation of the general concept of functionality depends."[1]

Summary. Some of the major objectives of mathematics teaching have not been acquired and cannot be attained by pupils as long as the great ideas are taught as isolated topics. Mastery of these ideas develops slowly. They must be presented repeatedly in various situations and at different levels. Each teacher must know what has preceded his course and build on it. Each must be familiar with that which is to follow and must pave the way for it. The gap between junior and senior high school mathematics will thus be eliminated. The accomplishment of these results requires the fullest coöperation of the teachers of both departments.

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[1] The National Committee on Mathematical Requirements, *The Reorganization of Mathematics in Secondary Education*, p. 12.

A SUMMARY OF SOME SCIENTIFIC INVESTIGATIONS OF THE TEACHING OF HIGH SCHOOL MATHEMATICS

BY H. E. BENZ
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I. INTRODUCTION

The research in this chapter received its first inspiration from a conviction on the part of the writer that a summary of scientific investigations relating to the teaching of high school mathematics should be made, similar to those for other school subjects.

The idea of such a summary persisted largely because the writer believed that less scientific research was being done in this field than the importance of the subject warranted; that the National Council of Teachers of Mathematics as an organization ought to be more interested in encouraging such research; and that any program of investigation must necessarily begin with a survey of what has already been done and an evaluation of the adequacy of this past research for curriculum-building purposes. As a result of definite encouragement from the editor of the *Yearbook*, the present study was made.

The work done here represents an effort to summarize the conclusions found in reports of scientific investigations of the teaching of high school mathematics. It is not claimed that the field has been thoroughly covered. Only those magazines which were found on the shelves of Ohio University were thoroughly searched. It is quite likely that several important researches were overlooked because they were reported in other magazines. No effort was made to survey systematically the literature written prior to 1915, although some earlier articles were included. Since the work was begun early in 1932, no articles for that year are included. It was at first hoped that unpublished dissertations submitted in partial fulfillment of the requirements for graduate degrees could be included, but this was found to be impossible. There is, of course, a

wealth of material in this source which should be included in subsequent summaries of this kind.

It was necessary to set up a more or less arbitrary criterion of research when selecting articles to be included in this report. The literature on the teaching of mathematics is very extensive, and a very large part of it consists of discussions of the topic which, although exceedingly helpful, were not, it will be noted, included in this summary. The reasoned conclusions of experienced teachers based on years of actual classroom contacts are not to be dismissed lightly, yet most articles based on such experience were not included. Descriptions of classroom procedures, written by critical, trained, and successful practitioners of the art of teaching, are often more helpful in pointing the way to improvement in methodology than the statistically reached results of a thoroughly scientific experiment carried out under elaborate and carefully controlled laboratory conditions. But this report does not claim to be a summary of such articles. Articles which represent merely opinions, or articles which describe the experience of an individual teacher with a certain technique, were not included regardless of the expertness of the opinion. There is no implication that such discussions are not helpful, and a subsequent analysis of the literature might include them, but the present discussion is confined to those articles which meet the commonly accepted requirements of scientific research in education.

The summary necessarily has been brief. It might have been more desirable, had conditions permitted, to discuss the articles in somewhat greater detail. The reader could judge better the validity of the conclusions if he could know the conditions under which the experiment was carried out. Obviously these conditions could not be described in sufficient detail to make this possible. The worker who is interested will wish to read the original articles in many cases. In some instances this is imperative, for some of them were of such a nature as to make it inadvisable to present even the findings. The reader is simply informed of the existence of the article, is told of its general nature, and is referred to the place in which the complete study can be found.

The setting up of a classification was somewhat difficult and the assignment of some of the reports to certain headings may seem arbitrary, but this plan was followed because some articles were found to fit with equal logic under several headings. If this account

seems to the reader to be somewhat disconnected, he should reflect that it is largely due to conditions inherent in the nature of the situation. It is to be hoped that some day we shall be able to write a systematic discussion of the teaching of high school mathematics from the results of more complete research. With the scientific study of the problem of high school teaching in its present state, it is impossible to write an account of the completed research with any semblance of system. This lack of scientific evidence was noted nearly twenty years ago [129],[1] and still persists to some extent.

II. The Mathematics Curriculum

Uses of mathematics in life. Studies dealing with the uses of elementary school arithmetic in everyday life have been frequent, but fewer investigations of the uses of high school mathematics have been made. This may be the result of the belief that many phases of high school mathematics have justifications other than their social utility. Only one investigation was found which dealt with the social utility of arithmetic as taught in the junior high school. Camerer [17] submitted to thirty-five bank employees and also to certain parents of school children a list of forty-one questions about those aspects of banking which are of common interest, asking their opinion on the relative importance of the items. The list of questions is given in the article, together with the order of their importance. Thorndike [113] made a study of the uses which people make of mathematics above arithmetic. In his study 200 pages from the beginning of each volume from one to twenty-eight of the *Encyclopædia Britannica* were read. A table shows the number of articles and the number of inches which contain one or more of several types of mathematical material. It was found that 3.57 per cent of the articles used mathematics beyond arithmetic, and these articles used 18.55 per cent of the space. The author concludes that "algebra is a useful subject, but its utility varies enormously." In a study carried on by Bobbitt and Scarf [10], the mathematics required for an understanding reading of popular science was investigated. A random sampling was made of a total of thirty-one issues of five magazines (three popular science magazines, one magazine with a science department, and one of general interest), and three popular science books. The report of the in-

[1] Numbers in brackets refer to titles in the Bibliography at the end of the chapter.

vestigation includes elaborate tabulations of units, measures, dates, processes, and kinds of numbers used in arithmetic. Twelve algebraic terms were found fifty-two times, and eighteen of these cases were instances of the use of the word "formula." Algebra was counted sixty-seven times, and seven different formulas were used. Negative or fractional exponents were discovered four times. A considerable number of graphs were found. The authors conclude, "Algebra is very little used by writers of popular science." On the other hand, geometric terms were used 5066 times. In addition to this, geometric principles, problems, and constructions occurred with great frequency. There were 196 cases of terms taken from mathematics above the level of that usually taught in the high school. Another report of the mathematics used in popular reading was given by Touton [115]. One of his students checked the uses of mathematics in a number of magazines and newspapers. Many concepts of form and quantity are listed in the report. In one popular magazine there was an average of forty-two mathematical concepts per page. "The concepts encountered by the general reader cover a very broad field."

Propædeutic uses of mathematics. Mathematics has frequently been urged as a suitable field for study in the high school because of the very great likelihood that the student will find uses for it in subsequent academic work. Several excellent studies of this relationship have been reported. Zerbe [131] investigated the geometry that is needed in high school physics. Ten physics texts and six laboratory manuals were examined. The results list twenty-five theorems and ten constructions that are involved in the geometry which the physics student should know. Rugg and Clark [92] in their very important study of the ninth grade mathematics curriculum give an analysis of algebra textbooks which shows how often the various operations of algebra were used (1) in subsequent topics in algebra, (2) in plane and solid geometry, (3) in advanced algebra, and (4) in physics and chemistry. They conclude that "half the material of first year algebra cannot be defended in terms of academic use." Thorndike [113] discusses an examination of forty-four high school texts in the social and physical sciences and the practical arts. The number of times that algebra was used and the number of inches of algebra in each text are noted. A table is given that shows which kind of algebra was used. The most extensive need of algebra in high school courses is said to arise in

connection with the reading of statistical graphs; in physics the student is discovered to require a mastery of formulas including an understanding of the manipulation which is implied; in chemistry he is found to need especially the ability to form proportions. Thorndike studied also the uses of algebra in college subjects. College teachers were asked to judge the importance of various topics, and a great difference of opinion was evident. Topics are listed, together with various methods of weighing the judgments obtained. Statistics and graphs stand high on the list, while literal equations other than formulas are thought to have low utility. In his investigation Thorndike checked the complexity of each topic by submitting various levels of algebraic tasks to college instructors of science, with the question, "Which of these abilities are essential?" Many insisted that the pupil must be able to perform algebraic operations on very difficult levels. It seems likely that what these college instructors really want is the degree of intelligence represented by the ability to do the tasks. Verbal problems were submitted with the request that the college instructors indicate which they considered important. Teachers of physical sciences rated the fantastic problems above the genuine. Teachers of social sciences, in general, rated verbal problems as useless.

Congdon [23] examined certain college texts to learn what mathematics is prerequisite. He listed certain concepts which should receive more practice in the high school, concluding that "college subjects dealing with quantitative data make use of an extensive symbolic language quite different from that used in high school mathematics." His report states that many special symbols are used, as well as subscripts and primes. Certain arithmetical computations more complicated than those usually found in elementary arithmetic are found in college physics. The geometry needed is very elementary. Scale drawings, line graphs, standard numbers, and the meanings of the trigonometric functions are used. In general, the mathematics needed is not difficult but often unusual. The mathematics involved in solving high school physics problems was also studied by Reagan [81]. He examined carefully the problem content of one text and found two hundred forty-one problems. The mathematical needs, he says, are not extensive, but they possibly cannot be met in certain cases because of what he calls "misplaced" emphasis in the mathematics courses. Few problems require any knowledge of geometry. In algebra more attention

should be given to formulas, ratio and proportion, and variation - topics which, as a general rule, are deferred too long in the traditional course.

The mathematics needed in the study of chemistry has received some attention. Rendahl [86] examined three high school chemistry books. He found 396 numerical problems, and counted the number of times that each arithmetical process was used and what fractions were involved. Decimals, percentage, and proportions were found frequently, although negative numbers and measurements were used very seldom. The uses of geometry he reports as negligible, while "the more important uses of algebra are in solving simple and fractional equations of the first degree with one unknown and in substituting in a formula." Williams [123] examined a text in college chemistry. He found "a rather limited use of mathematics." A thorough knowledge of the fundamental processes of arithmetic was seen to be necessary and a mastery of simple equations in one unknown and simple fractional equations to be valuable. "A knowledge of the fundamental processes in algebra seems desirable though by no means as essential as might be inferred." Zerbe [132] studied the relationship between geometry and plane trigonometry. In five trigonometry textbooks he found that sixty-eight geometric principles, theorems, corollaries, and constructions were used.

Undoubtedly, the common requirement of mathematics for college entrance may be traced at least partially to a conviction on the part of someone that mathematics is necessary for success in college. A study by Lewis [57] shows that from 1896 to 1916 the number of American schools requiring solid geometry for entrance decreased from thirteen to three.

Interest in mathematics. Pupils' interest in mathematics has been studied by several individuals. Downey [33] sent a questionnaire to 7000 pupils in fifteen high schools. Eighty-four per cent replied affirmatively to the question, "Do you like mathematics?" Liking is based mainly on the interest, usefulness, and mental training involved. In algebra it was found that manipulation and applications are equally popular. In geometry constructions are popular, while originals are unpopular. Thorndike [113] asked 1,300 high school pupils which subjects they liked most and least. For boys algebra holds a middle position, while girls tend to dislike this subject. About one-fourth of all girls reporting said they like it least

or next least of all the subjects taken in the high school. Plane geometry ranks below algebra for both groups. Thorndike also submitted various algebraic tasks to pupils for rating in terms of interest. He found that pupils do not prefer applied problems to abstract computations. They prefer numerical computations to literal computations, and dislike fractions and elaborate simplifications, although they like evaluation by substitution and the graphing of equations. Difficult material is unpopular. An interesting study is reported by Davis [27]. He asked business and professional men, who were prominent in their occupations, their opinions relative to the teaching of mathematics in the high school. He found an overwhelming opinion in favor of teaching algebra and geometry. Most of them claimed that a certain mental training had been received and a large majority insisted that mathematics must be retained as a school subject. In one city 91 per cent of the business and professional men recommended that mathematics be made a high school requirement. In a St. Louis high school about 60 per cent of the pupils claimed to like mathematics very much. The results from other questionnaires are cited to show that pupils enjoy mathematics, consider it valuable, and prefer it to other subjects.

Disciplinary values of mathematics. Most of the controversy on the question of whether or not mathematics should be a required subject in the high school has centered around the question of whether or not certain abilities, which it is believed are developed in connection with the study of mathematics, transfer to other fields. It is unfortunate that no more work has been done in order to ascertain the extent to which this transfer takes place. Rugg's study [91, 93], which he made by an analysis of 413 of the students at the University of Illinois, indicates that transfer is greatest in those fields most closely related to the subject in which the training is received. This study has to do with the disciplinary value of descriptive geometry. "The training operates more effectively in the case of problems containing content resembling that of the training course." Lewis [58] compared the marks of students in mathematics with ability in reasoning "to find if there is a reasoning faculty which by exercise in mathematics can be made stronger for other kinds of reasoning." The tests used were not objective and perhaps not very reliable, but the conclusion drawn is as follows: "These tests are surely convincing of one thing, viz., 'that students able in

mathematical reasoning are not even generally able in practical reasoning and law.'" Rietz [87] computed the r from Lewis' data and found it to be .27 in the case of law, which he believes indicates that Lewis' conclusion is not justified. To test these conclusions further, Rietz computed r's between the marks in mathematics and those in law and found that they range from .38 to .68. No partial correlations are given to show the relationship when the common factor of intelligence is held constant.

Kelley [54] carried on a study to determine the more general values of algebra. He used equated groups, one group taking the regular algebra course, the other taking a course designed to achieve certain broader values inherent in the subject. At the end of the year the pupils in the two sections showed little difference in their mastery of the algebraic method. The first group showed greater power of mathematical analysis, while the second group had achieved the broader values to a greater extent. The author concludes that "the varied and important values claimed for high school algebra may largely be realized under definitely aimed instruction."

Objectives. The determination of objectives would seem to be fundamental to any intelligent program of reorganization of mathematics. The method to be followed in determining objectives was the object of one study. Georges [39] asked forty-two teachers which of five procedures for determining objectives they favored, and found that most of them preferred following reports like that of the National Committee on the Reorganization of Mathematics, and the findings and conclusions of investigators in the field of objectives.

Schorling [98] made a comprehensive study of the entire field of junior high school mathematics in order to set up a tentative list of objectives. First he established a series of guiding principles and then determined by means of tests the mathematical equipment of beginning seventh grade pupils. His list of objectives was set up on the basis of social utility, courses of study, textbooks, the report of the National Committee, and the opinion of a carefully selected jury. These objectives were then used as the basis of a teaching program. The new material was taught under experimental conditions, with control sections using the regular course of study and regular material. Twenty-eight of thirty-four experimental classes did as well as or better than the corresponding control

classes. The material was then again revised in the light of the teachers' comments.

Considerable attention was devoted by Smith and Reeve to the question of objectives for the junior high school. Their rather comprehensive list [103], grouped under several headings, will be found useful for checking courses of study or textbooks. Lists of objectives in geometry [83] and intermediate algebra [84] were worked out by Reeve with the assistance of a "large number . . . of experienced teachers of mathematics throughout the country." Nyberg [69] set up a list of objectives based on accurate records of classroom procedure. He listed for thirteen classes just what happened every day. Nyberg says that Reeve's list is too inclusive and suggests sixteen topics which he cannot find time to teach.

Miscellaneous studies on the curriculum. Johnson [52] investigated the teaching of geometric material in the junior high school. He asked twenty-four members of the Chicago Men's Mathematics Club about the grade placement of 125 items in geometry, and whether they should be taught at all. There was a wide variation of opinion. He examined the geometric and trigonometric material in ten series of junior high school texts, and again found a wide variation of opinion. No one, not even writers of texts, seems to know what ought to be taught and in what year the material should be placed.

Rugg and Clark [92] investigated the question of "thinking" in the algebra course, and from an examination of textbooks concluded that less than one-third of the time in algebra is devoted to problems requiring thinking. Many high schools find themselves faced with the necessity of deciding which of several courses to offer. Babcock [7] made an investigation to determine whether solid geometry or advanced algebra should be offered in a coeducational secondary school which cannot offer both. Sixty heads of departments were asked which of the two was of greater value. The majority favored offering solid geometry.

III. The History and Status of Mathematics Teaching

History. Mathematics is sometimes thought to be a relatively static subject. Its teaching, however, has changed during the past few years. A comprehensive study of the history of the teaching of elementary geometry from the time of ancient Greece to 1909 was carried on by Stamper [105]. In his report he discusses also

the current teaching of geometry in the various countries of Europe, and an effort is made to interpret contemporary problems in the light of their historical background. Chateauneuf [19] traced changes in the content of elementary algebra since the beginning of the high school period. She gives tables and charts showing changes in emphasis, and the number and percentage of exercises devoted to various topics in succeeding decades. General trends and significant changes are discussed with some care. Three major changes have come about in the teaching of algebra in the last hundred years: the introduction of graphs about 1890, the elimination of progressions, and the introduction of trigonometric ratios. Minor changes include a gradual increase in the number of exercises, largely in verbal problems, and a reaction against complexity. In general, algebra is changing in the direction recommended by the National Committee in 1923 (in many instances this trend was noticeable as far back as 1900). Sanford [95], in a careful study of the history of algebra problems, followed various types of problems from their first appearance, in some cases hundreds of years ago, to the present. Simons [102] traced the history of the introduction of algebra into American schools by means of a study of early manuscripts, students' notebooks, commencement theses, textbooks, school records, and advertisements of schools which appeared in the newspapers. "Nowhere are there found indications that a practical need for algebra actuated the teaching of it during the early period." It was evidently taught for its own sake. Seybolt [101] discusses briefly the teaching of mathematics in the colleges and secondary schools of America in the eighteenth century. He gives many footnote references to the newspapers of the time.

Current practice. Many studies have been made which throw light on present procedure in the teaching of mathematics in the United States. Some of these are referred to under other headings, because they seem to have some other purpose than the determination of the present state of affairs. Current practice in the teaching of junior high school mathematics was investigated by Worthington [128]. His questionnaire study of eighteen junior high schools in Pennsylvania and thirty-three outside Pennsylvania showed the time devoted to the subject and the texts used in each grade. Usually four or five periods per week were devoted to mathematics Only one text was used in as many as ten schools. The fifty-one schools used sixteen different texts in the seventh grade. Almost

half the schools taught trigonometry. About 30 per cent taught some demonstrative geometry, while logarithms and the use of the slide rule were rarely given. In an article entitled "Hotz Algebra Scales in the Pacific Northwest," Eells [34] reports the scores of more than 4000 students in eighty-nine schools. Detailed results are given and schools are classified according to the text used and according to the size of the school. No statistical analysis is made to show whether differences between schools using different texts are significant. Seventy-five per cent of the schools had medians which were above the test norms. Rosenberger [89] investigated the place of elementary calculus in the senior high school program. He describes foreign practice, gives a comprehensive history of the development of calculus as a school study, and then sets up a proposed syllabus for use in the secondary schools.

IV. The Organization of Subject Matter

Correlated and general mathematics. Perhaps the leading proponent of correlated mathematics in this country is Breslich. In a report [15] he gives the results of a study of conditions at the University of Chicago High School. As a result of the development of correlated mathematics, failures were reduced, until in 1920 they were less than the percentage of failures throughout the school as a whole. Withdrawals during the semester decreased, and the number of pupils electing third and fourth year mathematics increased. Students who later took college mathematics at the University of Chicago made an average number of grade points which was above the class average, although they did not do the same above-average work in other subjects. Of those who went to other colleges, over half did better in mathematics than in other subjects.

Burks [16] compared students who took general mathematics with those who took algebra and geometry. The students were paired on the basis of past work and intelligence. The general mathematics group learned as much or more mathematics in seventy weeks as the algebra-geometry group learned in seventy-six weeks. Forty-five per cent of the general mathematics group elected advanced algebra and 24 per cent elected solid geometry, while 15 per cent of the algebra-geometry group elected advanced algebra and 14 per cent elected solid geometry. General mathematics seems to be more interesting. Crow and Dvorak [25] gave the Hotz Algebra Scales and the Minnick Geometry Tests to pupils

in five high schools. Some had taken general mathematics and some had taken algebra and geometry. The different groups were not in the same school and groups were not paired and not equal. Intelligence scores were not available. On the algebra test the general mathematics group was superior, but not by a significant amount. On the four parts of the geometry test the general mathematics group was superior and the chances that the differences were real were 38 to 1, 160 to 1, 369 to 1, and 6.5 to 1, respectively. The authors conclude that "the experiment tends to show that the reorganized mathematics gives the pupil as much knowledge as the old type of organization, though no decided advantage." Insofar as geometry is concerned, the authors seem to be very conservative in their conclusions. Stokes [108] examined the ninth grade records of pupils who had had arithmetic in Grades 7 and 8 and those who had had general mathematics. The ninth grade course was general mathematics. Those who had had general mathematics in the seventh and eighth grades did better work in the ninth grade by an amount which was 8.29 times the standard error of the difference. The chances are more than 999 in 1000 that the difference is real. Seventh and eighth grade general mathematics is a better preparation for ninth grade mathematics than arithmetic. Wallace [118] compared the work of two equated ninth grade classes, one of which was taught "correlated" mathematics and the other "unified" mathematics. In his report the author explains the difference between these two terms. No statistically significant difference in performance was found between the two classes. The cumulative findings indicate that any difference there may be is in favor of unified mathematics. A most thorough and extensive study of general mathematics is reported by McCormick [59]. He gives a history of the movement, cites evidence bearing on the results of teaching general mathematics, and gives the results of an analysis of textbooks used in the high school. His study concludes with a series of concrete suggestions for teaching. Orleans [71] describes an experiment in teaching advanced algebra and trigonometry as a fused course in Grade 11. He notes the results of Regents' examinations. The results are not very conclusive because of the presence of certain uncontrolled factors and the unreliability of the tests used, but they indicate that pupils learned as much algebra and as much trigonometry as they do when the courses are given separately.

The analysis of algebra. A most complete analysis of the fundamental skills of algebra was made by Everett [37]. He identifies "associative" skills as contrasted with "manipulative" skills. The former are those which "give meanings to operations and relations." These associative skills, their place in the subject, and their importance are described and discussed fully. Errors which pupils make because of a lack of possession of the associative skills are described in some detail. Pease [74] made a thorough and careful analysis of the processes to be taught and the bonds to be formed in elementary algebra, and determined the relative difficulty of each process by determining the number of wrong answers which pupils gave to certain test items. He found that algebra involves 541 different learning units. He notes the complexity of the mental functions involved and points out the need of an analysis of these functions. Waples and Stone [119] made an elaborate analysis from the teaching standpoint of a single topic in algebra—directed numbers. The unit was analyzed for desired outcomes, and pupils' difficulties were then determined and classified. The purpose of the study was to "define a technique whereby any investigator may collect methods of removing pupils' difficulties in learning a given unit."

The analysis of geometry. The common subject matter of geometry as represented in six textbooks was analyzed by Welte [122]. He determined which propositions are common to the various books and to the list proposed by the National Committee. He listed the technical vocabulary of geometry, the "main ideas" of geometry, and the frequency with which each word and each idea is used. He then set up a "psychological photograph," a table showing which words and which ideas are used in each theorem.

V. Conditions Affecting or Accompanying Teaching

Mathematical ability. The analysis of mathematical abilities has proved to be a fruitful field for investigation. Several writers have carried on studies of the relationship between ability in mathematics and other mental abilities. McCoy [60] found ability in algebra related to the following to the extent represented by the coefficients given: arithmetic, .59; reading, .32; intelligence, .48; chronological age, .12; interest and persistence, .40.

Thorndike [113] quotes a study by Weglein in which the r between algebra and the average of all first year studies in the high

school including algebra is .64. Crathorne found an *r* of .50 between algebra and intelligence, Buckingham found .38 between algebra and scores on the Army Alpha test, and Proctor found .46 between algebra and scores on the Stanford-Binet test. Thorndike quotes several other studies and the coefficients of correlation which were found. He points out that these correlations are higher when corrected for attenuation, and still higher when corrected to express the relationship as it would be in a standard, unselected, adult group, when the *r*'s run from .74 to .92. Buckingham [15] examined the records of twenty-seven pupils who failed in algebra in the Champaign High School. The *r* between scores on the Otis test and scores on the Rogers mathematical prognosis test was .32. Buckingham quotes several other investigations as follows: Jordan at Arkansas found that the *r* between Army Alpha and mathematics grades was .21 for ninety-four students, smaller than the relationship between Army Alpha and any other subject investigated. At the Harrison Technical High School in Chicago the *r* between the Terman group tests and algebra grades was .25 for 235 cases. At the Urbana, Illinois, High School the *r* between Army Alpha and algebra marks was .38 for 195 cases, and that between Army Alpha and geometry marks was .40 for the same group. These facts lead to the conclusion that mathematical ability is not so closely related to general intelligence as has been supposed.

An elaborate study of the factors of success in first-year algebra was made by Schreiber [99]. The Courtis test, some of the Hotz tests, and the Otis intelligence test were given to 160 pupils during the last month of school. Schreiber reports that the algebra pupils were above normal in arithmetic ability, but that the relationship between algebra and arithmetic was low when intelligence was held constant. The correlation between algebra and intelligence was higher than the correlation between algebra and arithmetic. Those who failed in algebra were below standard in arithmetic and intelligence. Schreiber concludes that an I.Q. of 90 is the minimum necessary for success in algebra.

A study by Haertter [44] was carried on to determine the knowledge which pupils brought to their tenth grade mathematics work. The test used is described in detail and one of the tests is given. He found that the pupils tested brought a large store of geometric information, and that it is not necessary to spend several weeks teaching terms and concepts if pupils have had the reor-

ganized program in the junior high school, thus permitting the covering of more work in the senior high school course. Boyce [11] carried on an investigation with a class in algebra, giving them tests in intelligence and arithmetic at the beginning of the year, part of the Hotz tests at the end of three months, and the same after six months of algebra study. Correlations between the algebra test and various parts of the arithmetic test were low. "When a student has passed through the arithmetic of Grades 1 to 8 inclusive, success in the first three months of algebra seems but very slightly dependent upon previous attainment in arithmetic." Lee and Lee [56] made a statistical analysis of certain data dealing with the performance of pupils on prognosis and achievement tests in algebra and geometry and made a comparison with previous investigations. They also studied algebra and geometry marks. Their conclusions follow: Algebra ability and geometry ability are probably related to an extent best represented by an r between .50 and .65. The r between achievement in algebra and achievement in geometry is probably between .40 and .70. Correlations between abilities in the two subjects are usually higher and more consistent than correlations between achievements in the two subjects. About 40 per cent of the pupils show a difference between algebra and geometry in respect to both ability and achievement that cannot be attributed to chance. Some factor other than achievement and ability is entering into school marks in these two subjects. In addition, the authors state several less important conclusions. It is unfortunate that this interesting and significant study did not involve more cases, but only two classes were used.

Attempts to develop prognostic techniques of mathematical ability are reported by two investigators. Perry [78] made a comparison of the prognostic value of the Orleans Geometry Prognosis Test, the Perry geometry prognostic test, and the intelligence quotient as determined by the Terman group test. Achievement was measured by the Hart Geometry Tests, Test I, and by teachers' marks. The Orleans test is more closely correlated with intelligence and it predicts performance in geometry better than the Perry test, but when the I.Q. is partialled out, the Perry test predicts better. The highest r found is .78 between the Orleans test and the Hart geometry test. This is a zero order r. One important difference between the two tests is noted: The Orleans test measures students' performance in the face of actual geometric situations, while the

Perry test tries to give the "potential abilities which, when directed, can lead to an effective mastery of plane geometry." Rogers [88] set up seventeen tests of mathematical aptitude and investigated their value as prognostic of success. She determined the relationship by the performance of pupils on each test and by general ability. The six best tests were selected for use in a composite prognostic test. "The correspondence found between the mathematical abilities tested might be traced to the common characteristic of capacity to react to partial elements in the situation." "Mathematical ability can be satisfactorily diagnosed by six tests requiring an hour and a half in time." In a total of 114 cases, the composite diagnostic test predicted school marks to the extent represented by *r*'s of .79 and .92 in two schools, when the obtained *r*'s were corrected for attenuation.

Individual differences. Differences in the abilities of pupils to learn mathematics have interested workers in this field for some time. Probably because of the relative ease with which studies of differences can be carried out, this problem has been carefully investigated. First, with reference to sex differences, Thorndike [113] reports that "on the whole the ability of the sexes is on a par." He finds also that boys in Grade 12 place algebra higher in the order of school subjects for preference than girls do. Pease [75] examined tests taken by 685 pupils and counted the errors in each of twenty-three fields of algebra. When the results are expressed in terms of the number of errors per hundred pupils, he finds that boys make more errors in twenty-two of the twenty-three skills measured. The average boy makes 164 errors and the average girl 143. Webb [121], making an investigation in geometry, tested 1130 pupils, classified into five groups according to mental age. In general, boys are found to be superior, and this superiority is most marked at lower mental ages. Only in the group with mental ages of 18½ and over are the girls superior to the boys. Girls, in general, attain one more year of mental maturity than boys before their achievement in geometry is comparable to that of the boys. When the middle 68 per cent of cases is considered, girls are found to be more variable. In his study of New York Regents' papers for 2800 pupils in 100 schools, Touton [116] found that the boys showed decidedly stronger preference for construction exercises than girls did, but the range within either sex was much more significant than the difference between the sexes. Perry [76] followed the progress

of two geometric sections, an experimental and a controlled group. The boys were higher in mental ability and reasoning ability, but the girls reached a higher level of achievement in the geometric exercises. No statistical analysis of the results is indicated in the report. Taylor [112] kept a careful check of the daily work of seventy algebra pupils. A record was kept of the time spent and the number of items correct on certain daily drill exercises. He found that those who got the most right answers did their work in the least time. In general, the brighter pupils wasted that part of the class period which was devoted to remedying the deficiencies of their less fortunate fellows. Taylor concludes that pupils of widely varying abilities should not be placed in the same class. The ability of students to profit by the study of algebra was carefully studied by Symonds [110]. In connection with his results Symonds gives suggestions for revision of the course of study in algebra.

Ability grouping. The problem of the homogeneous grouping of students has occupied the attention of educators for some time, and the merits of the plan have been carefully investigated. Several studies have applied to the field of mathematics. Kerr [55] reports an experiment carried on in one of the high schools of Cleveland in which pupils were classified in mathematics according to ability, slow groups taking courses in industrial mathematics with much drill and much review of arithmetic in the ninth grade, and taking a tenth grade course consisting of material geometric in nature but without demonstration. Constructions and applications were stressed. Under this scheme failures were reduced by about one-half, pupils were found to be happier, and teachers better satisfied. In a systematic consideration of the problem of homogeneous grouping, Mensenkamp [62] discusses the various bases for sectioning and describes how the work may be adjusted to the needs of varying groups. He gives correlations between different predictive measures and success, and concludes that those in the lowest fifth in the eighth grade should take no algebra at all. In the experiment described, the remainder were sectioned on the basis of eighth grade arithmetic scores and intelligence as determined by the Otis test. Austin [6] reports an experiment in which pupils were classified by the grade teachers into A, B, and C groups. About 15 per cent were thus assigned incorrectly. "The Otis test would have placed them all correctly." Intelligence, he says, is a necessary but not a

sufficient condition for success in algebra. Any pupil with an I.B. (Otis) of 100 can pass in algebra if he is willing to make the effort, but new material and new techniques must be developed for the use of the slower sections. In another place [5] the same author reports that the Otis test is superior to an arithmetic reasoning test for sectioning algebra pupils, although the Otis test seems to be somewhat less than perfect, for, he says, the r's between the I.B. and algebra marks were found to range from .40 to .54.

Class size. The optimum size of high school mathematics classes was the subject of two investigations. Jensen [50] reports a very carefully controlled experiment in which pupils were put into large classes and small classes. The individuals were paired by I.Q., age, and initial algebra ability. Other factors were carefully controlled. The progress of the two groups was then compared. He concludes that achievement in elementary algebra is more rapid in small than in large classes, and the difference is more marked in the case of boys. Jensen used only forty cases, and his differences in favor of small classes, although he draws the above conclusion without reservation, do not seem to warrant considering the question closed. Haertter [43] placed twenty pupils in a small class in geometry and fifty-five in a large one. Each pupil in the small class was paired with two in the large class. The two groups were about equal in average I.Q., average marks in the previous year, and average mental age. He describes in detail the technique used in the large group, which involved breaking it up into small groups. Pupil leaders assisted in carrying out administrative duties. At the end of the year the small section had a median score of 562 and the forty paired students of the large section a median score of 552.5. The comparison of the individual scores indicated that the poorer students did better work in the smaller section, but neither one of the sections was found to be demonstrably superior for the better pupils.

Study. Individual and group study habits seem to have some effect on the extent to which pupils master mathematics in the high school. Stokes [109] describes a technique for measuring application in a mathematics class. Observers used stop watches to time individual pupils in a group to determine the exact number of minutes they were studying. The distribution for the ninety-two pupils was far from normal, but the average index of application increased during the year. In general pupils worked about 82 per

cent of the time. The correlations indicate that application has more effect on achievement than intelligence has.

Johnson [53] compared classes which used the traditional question and answer method with those which used a socialized project stu... method. The classes were about equal in ability, but in the case of both first and second year groups, the socialized recitation pupils had better scores on each of four quarterly tests and the semester examinations. No statistical analysis of the significance of the difference is given. Douglass [29, 32] compared the study-recite sequence with the recite-study sequence in several subjects including mathematics. "The R-S sequence is probably more effective for classes in mathematics in the junior high school." In the three classes used for experiment the chances in 100 that the differences were real were 97, 56, and 88, respectively.

Minnick [63] observed the performance of thirty-six pupils divided into two groups at random. One group spent the recitation period in geometry and did their study without supervision. The other spent a period directly following the recitation period in preparing the next day's work. The supervised group kept notebooks. "An effort was made to teach the child how to study geometry." "In each of the six week's examinations and the final examination, the supervised class excelled in both the average grade and the average number of problems solved." No statistical analysis of the differences is given. Experiments in a number of schools with several subjects are reported by Brown and Worthington [14]. Supervised study groups had sixty-minute periods and other groups had forty-five-minute periods. In one school the algebra classes showed no differences. In a second school the supervised study group showed "marked superiority" by all measures of progress. The groups were equal in intelligence and at the end of the semester the median semester marks were 76.8 and 70.0 for the supervised study group and the recitation group, respectively. The probable error of this difference is not given.

Barton [8] equated two groups on the basis of chronological age, I.Q., and ability as measured by the Stevenson Problem Analysis Test. There were eleven pupils in each group. Group A used a class discussion method for solving problems, while Group B used a technique by which individuals received specific assignments. The group discussion method was found to be somewhat superior.

Teachers of mathematics. A comprehensive study of teachers' subject-matter combinations was made by Anderson and Eliassen [1]. They covered twenty-five different geographical areas in the United States, some of which included whole states. Seventeen per cent of all the teachers listed taught one or more classes in mathematics. In West Virginia only 5 per cent participated in instruction in this field, while in North Dakota the percentage was 37. Of 673 teachers in North Dakota, 246 taught some mathematics and only eight taught only mathematics. Forty-five per cent of the mathematics teachers in New York State taught only mathematics. Science was the subject most commonly combined with mathematics in teachers' programs.

In a study of the training of mathematics teachers, Zant [130] describes the mathematics programs of schools in England and Germany, the training of teachers, objectives, standards, results obtained, etc. Standards are much higher in both these countries than in the United States, but in this country more students take the work. It was found that teacher-training requirements are higher in Europe. In an article by Hughes [48] a proposal is made for a study intended to find out just how much college mathematics a teacher ought to have. Hughes suggests comparing the results obtained in terms of pupil mastery by teachers with varying amounts of preparation.

The problem of providing material for a teacher-training course was attacked by Schaaf [97]. He studied the needs of prospective teachers of junior high school mathematics, examined catalogue descriptions of courses offered to such prospective teachers, and gave tests to freshmen enrolled in teachers colleges. He then set forth in some detail the material needed by such individuals in the form of a syllabus for a proposed course to be taken by teachers in training.

VI. The Learning of Mathematics

Drill. Those aspects of teaching that are most objective are likely to receive the most attention from experimenters in education. The care with which the psychology of drill has been investigated illustrates this fact. The amount of drill needed to master algebraic tasks was studied by Thorndike [113]. Sixty-eight teachers of mathematics estimated how many times a pupil in a typical ninth grade class would do each of thirty tasks. Thorndike gives these

tasks, together with the number of times estimated. Estimates of how often pupils represent a number by a letter varied from 100 to 50,000. Estimates of how often they add algebraic expressions with unlike signs varied from 100 to 1,000,000, and estimates of the number of times they divide by a fraction varied from 1. to 1000. A study of textbooks showed a variation from 367 to 559 for the first of these three tasks and from 30 to 62 for the last. "The plain fact is that nobody, not even an author of a textbook, has exact knowledge of the amount of practice it contains unless he actually makes the count."

The need of systematic drill is discussed at some length by Rugg and Clark [92]. These men describe the construction of suitable drill exercises and give instructions to indicate how they are to be used. Taggart [111] devised a set of drill-tests for algebra similar to the Courtis tests in arithmetic. Two sections used them and one section did not. The results indicate that such systematic drill-testing is worth while. Coit [22] describes an expe iment in which tests were given to 260 pupils in various classes in four schools. Subtraction was selected for remedial drill and was practiced ten minutes per day. In a retest the percentage of error dropped for every item. After two months without drill the test was given again and the effects of the drill were still evident, for the percentages of error were even lower. In three high schools the drill was not given and the reduction in percentages of error was much less. In a somewhat similar study made to determine the effect of drill, Armstrong [2] used comparable groups in algebra, and the experimental group was given an eight-minute drill three to five times per week for seven months. The groups were paired for intelligence. At the end of that time the experimental group showed a superior score on the Hotz tests by an amount which indicates that the chances were 89 in 100 that the difference was real. The groups did not differ in problem-solving ability at the end of the experimental period, which fact may be explained by the designation given to the drills used, "Accuracy-Drill-Tests."

Retention. Intimately associated with the problem of the efficacy of drill is the one of retention. Worcester [127] gave different forms of the Douglass algebra test on February 9, March 29, and May 17 to a "small class." The tests were repeated the following December. On the retest of the form used in February the average was 1.09 of the first score. The December average for

the test given in March was .85 of the first score, while the retest of the third form, given in May, showed an average of .35 of the first score. Certain items were remembered by nearly 100 per cent of the class. Thorndike [113] gave a short algebra test to 189 college graduates, students in a law school. By estimating from typical high school performance what the individuals would have done at the time they finished algebra, he concludes that the commonly accepted notions about forgetting are not warranted. Eells [35] studied the amount of algebra retained by college freshmen. He discovered that they showed skill above the Hotz standards for problem solving, but below those standards for manipulation, unless their record showed that they had had three semesters of algebra in the high school. These were freshmen who elected college mathematics. Those with three semesters of entrance credit in algebra did not seem to exceed in problem-solving ability to any great extent those with only two semesters. The author refers to a study by Woody in which opposite results were obtained in one Michigan high school and quotes Woody's conclusion: "The seniors had retained a relatively large amount of the knowledge of the more formal aspects of algebra, but a comparatively small amount of the more complex problem aspects of the subject."

More forgetting is indicated in a study by Arnold [3] of the results obtained by testing fifty-two college students with Hotz tests. In four out of five scales the median for the college students was below that for the end of the first year of high school. About half the students had offered three semesters of high school algebra for entrance, and the test was given after they had completed one-third of the first year of college algebra. Evidently the first part of the college algebra course was not a review of high school algebra, as is frequently the case. Arnold remarks that the students tested showed extreme deficiency in all phases of algebra. Retention for a shorter length of time was studied by Mirick and Sanford [67]. Two tests were given, one at the very beginning of the eleventh year and one five weeks later, to students who had had two years of mathematics, some, one year each of algebra and of geometry, and some, two years of algebra. The results were disappointing to the experimenters, and they comment at some length about the undesirability of teaching many topics in algebra as though teachers meant merely to show pupils that "it can be done." Pupils achieve real mastery of only a very few simple tasks.

While the retention of algebra has been studied rather carefully, only one investigation in the field of geometry was found. Arnold [4] gave the Schorling-Sanford test to eighty college students and found that 86 per cent were below the standard median and 26 per cent were below the tenth percentile. "While the test disclosed a lack of knowledge of basic geometric facts, based largely on definitions, theorems, and formulae, the most outstanding difficulty appeared in the drawing of conclusions from given data."

Errors. An analysis of errors found on pupils' papers offers a relatively easy method of studying the teaching of algebra, but it must be admitted that in many cases it is difficult to draw appropriate conclusions from the facts which are found. One investigator studied the arithmetical errors made by high school pupils. Minnick [65] reports the results of arithmetic tests given to pupils entering the high school. The errors were classified and the types listed for each problem.

Errors in algebra were the subject of four investigations. Thorndike [113] gives tables to show the percentages of wrong responses to simple algebraic tasks. He finds that "pupils lack mastery of the elements of algebra." An interesting illustration of the kind of error study which a classroom teacher can make is found in an article by Wattawa [120]. She recorded all errors made over a period of three months and found that many were of an arithmetical kind. In home work this type constituted 32 per cent of the total. In class work it constituted 42 per cent of the total, while 41 per cent of the errors found in the pupils' work by the Hotz formula and equation scale were arithmetical in nature. A total of 407 errors were listed. An extended report by Benz [9] of the results of the Ohio Every-Pupil-Test in Algebra for 1930-31 gives the distribution for 16,927 papers. Three hundred typical papers were analyzed in detail and the types of errors were listed. Each item is discussed in the light of the errors most commonly found, and suggestions are made for teaching the principal topic involved. Pease [74] studied the errors made in certain algebraic processes at the time the topic was completed. He tabulates and classifies these by "error-types." In all, 43,828 errors are tabulated and classified. His summary table shows that sign errors constituted 23 per cent of the total and process errors 31 per cent. Pease found 227 different error-types scattered among twenty-three algebraic topics. The results of the Ohio Every-Pupil-Test in Geometry were

critically studied by Christofferson [20]. He examined typical papers and, in the light of the errors made, sets forth certain suggestions for teachers of geometry.

Failures. The literature on the teaching of high school mathematics indicates that much of the antagonism against making this subject a requirement grows out of the large number of failures. This phenomenon has been studied by many workers in the general field of secondary education. Special studies of failures in mathematics are reported by three individuals. Elder [36] applied the Indiana Mental Survey scales and the Otis Self Administering Tests to fifty pupils. The r between the mean percentile ranks on the two tests and scores in algebra was .60. Assuming that the upper three-fourths of the scores in algebra were satisfactory, he concludes that 89 per cent of those above the thirtieth percentile in intelligence will pass in algebra and only 31 per cent of those below this point will pass. Wood [125] made a careful study of a group of twenty-three pupils who had failed in algebra. When correlating their I.Q.'s and class grades, he found that R (Spearman) equaled .993. A comparison of the means of the Rugg-Clark tests and the I.Q.'s showed an R of .998, while the means of the Rugg-Clark tests and teachers' marks showed an R of .999. "The class grades were not based on results of the test." The author made out a verbal problem test on the basis of the text which the pupils had used in their algebra work and found that this correlated with the means of the Rugg-Clark tests to an extent represented by an R of .979. The conclusion is drawn that algebra is so closely related to mental ability that conducting classes for "repeaters" is a waste of time. Certain questions relative to the size of the coefficients of correlation shown will undoubtedly occur to the reader.

Causes of failure in plane geometry were studied by Crafts [24]. Intelligence, records, and study habits of a group of failing pupils were examined. It was concluded that 30 per cent of the failures were due to incapacity. Over 50 per cent of the failures were due to conditions which the school could remedy.

Various features of algebra learning. In a thorough discussion of the teaching of algebra, Rugg and Clark [92] describe a program of experimental teaching. They did not use control groups but selected new teaching procedures, then used them under the direction of a trained observer. Careful notes of difficulties encountered were made immediately after each lesson and these notes

were used as a guide for future teaching. They found that in some cases their procedure permitted the topic to be covered in less time than was usually the case. Comparative tests showed higher scores for the experimental groups than for other groups. It should be said that the teaching was so completely reorganized that it is difficult to determine to just which features the improved results were due.

An analysis of the difficulties involved in a specific topic in algebra was made by Dickinson and Ruch [28]. They studied the effect of subscripts, decimal coefficients, and upper case letters on difficulty in factoring. They gave eight tests involving 160 items to a total of 600 pupils. Three examples which involved subscripts were missed by 70, 56, and 62 per cent of the pupils, respectively, while parallel examples with this difficulty removed were missed by 13, 14, and 31 per cent, respectively. Examples involving decimal coefficients were missed by many more pupils than similar examples which did not involve this particular difficulty. Exercises involving upper case letters were missed more often than similar examples using lower case letters. While the various items compared were not exactly parallel, they were sufficiently similar to warrant the conclusion that the elements involved need further treatment if they are not to be sources of difficulty to pupils.

Jackson [49] studied the relationship between the I.Q. and success in algebra of a group of boys in a private preparatory school. He concludes that a boy with an I.Q. of 110 can learn algebra if he wants to.

Thorndike [113] studied the relationship of ability in the mechanical skills to intelligence. He concludes that "algebraic computation as actually found is emphatically an intellectual activity. . . . It is far above the reproach of being a mechanical routine which can be learned and operated without thought."

Various features of geometry learning. In a study of 441 applicants for admission to Cooper Union who took a geometry placement test and told how long they had studied the subject, Snedden [104] found an r of only .218 between success on the test and the length of time the subject had been studied. He concludes that "the amount of time a person has put on a subject is worth almost nothing as an index of his proficiency in that subject." Pitts and Davis [79] made a comparison of the analytic and synthetic methods of teaching geometry. They used two equivalent groups

and conclude that, when used as the only method of attack, the analytic method acts as "a hindrance, deterrent, and eliminator." The analytic method was not a help in the solution of originals. As the classes involved used an elaborate form for recording proofs, it is possible that this was a major factor in the experiment in both classes and actually rendered more difficult the thinking that the analytic method is supposed to encourage. The experiment should be repeated with greater attention paid to thinking and less to form.

The teaching of geometric concepts was studied by Scofield [100]. Four classes were used. Group A, C, and D were practically equal in intelligence, while Group B was superior. Groups A and B used the traditional method, while Groups C and D used an inductive method. No textbook was used, but figures were placed on the blackboard and discussed. At the end of three weeks a test was given on the mastery of the concepts. Groups C and D did better than Group A. This superiority carried over into the demonstration work later in the semester. "The experimental method has given the pupil clearer conceptions, keener interest in geometry, and a better approach to demonstrative work than has the traditional."

A study of the assignment in geometry is reported by Chastain [18]. In this experiment two sections were used and individual pupils were paired on the basis of intelligence. The experimental group used fifteen minutes each day in going over the assignment, "a carefully planned, analytical assignment." The pupils participated in outlining the exercises assigned. Geometry tests were given every two weeks and scores for the two groups were compared. The difference was found to be 7.9 times its standard error in favor of the experimental group. The author concludes that this "difference . . . cannot be caused by chance." Every effort seems to have been made to equalize the two groups. There were perhaps too few cases and the tests were not as objective as might be wished.

Modern authorities on the teaching of geometry insist that one of the important objectives is to teach children to think. The extent to which this can be done has not been sufficiently investigated. Johnson [51] studied this problem, using parallel groups. The experimental group frequently discussed common problems of everyday life which demanded the use of reasoning, and examined the logical processes used in their solution. The Burt Reasoning Test was given before and after the experiment, and a small gain in reasoning ability was found in favor of the experimental group.

Perry [77] made a careful analysis of the reasoning process and then devised a technique of reasoning suitable to demonstrative geometry. An experimental section was taught, the teacher using this technique, while control groups were taught by a more or less conventional method. In the experimental section difficulties were fewer and achievement was greater. The very superior students, however, were not helped by the experimental technique. The experimental group showed a marked improvement in the ability to do non-mathematical reasoning. The process of reasoning as used in the solution of original exercises in geometry was studied by Touton [117]. He examined the work done by pupils on this type of material on 2800 geometry papers written for the New York Regents' examinations. He formulated a list of the steps in the thought process which should be of benefit to teachers of mathematics.

VII. Problem Solving

Problem material. The place of verbal problems in the algebra curriculum has been the subject of some controversy. Problem material was carefully studied by Powell [80]. He presents a collection of 466 verbal problems rated by algebra teachers for genuineness, importance, and interest. His estimates of difficulty are based on pupil performance. The associations are shown between the various factors and suggestions are made relative to teaching procedures. Powell discovered that teachers are poor judges of pupils' interests, and that pupils are most interested in problems that imply activity and in puzzle problems when they are so stated. Reasonableness is a minor factor in the development and maintaining of interest, while difficulty tends to make problems less interesting.

Teaching problem solving. The difficulties involved in solving verbal problems were studied by Clem and Hendershot [21]. They examined the papers of eighty pupils in four classes and concluded that problem solving depends on problem reading; that the inability to do logical reasoning is a significant factor in producing errors; that many capable pupils fail because they are not taught to work systematically; and that many errors are caused by the pupils' lack of mastery of arithmetic and by failure to check. Certain suggestions designed to be helpful to teachers are included in their report.

Problem solving is closely related to reading ability. The difficulties encountered in reading mathematical material were studied by Georges [40] He used an individual interview technique and collected 188 reports. Difficulties which pupils encountered in reading problems for solution, descriptive material, and illustrative problems are listed. Thirty-seven per cent of the difficulties were in understanding and interpreting statements, because of pupils' unfamiliarity with mathematical vocabulary or mathematical symbolism. A lack of a "mathematical apperceptive mass" accounted for another 21 per cent of the difficulties. Many difficulties were caused by the presence of specific words or phrases. Those symbols, processes, and relationships which caused trouble are listed.

The eye-movements involved in reading formulas were studied by the photographic method by Tinker [114]. He discovered that formulas demand more fixations than ordinary prose and concluded that pupils should be taught how to read mathematical formulas.

VIII. Instructional Materials

Junior high school textbooks. Textbooks in junior high school mathematics have been the subject of three important analyses. Faller [38] describes the procedure of a committee selected to evaluate junior high school mathematics textbooks as a guide for a large city. The procedure is excellent, but such an elaborate study is impossible for most school systems. Committees were appointed to analyze textbooks from each of the following points of view: content; presentations, discussions, and methods; manner of development; drills, tests, summaries, reviews; vocabulary; illustrations; authors; size of numbers; mechanical phases.

Williams [124] examined and evaluated three series of junior high school texts in terms of standards set up by the North Central Association. Correlations were worked out between the percentage of space allotted to eighteen different divisions of subject matter and the percentage recommended. For the three series the *r*'s were .44, .52, and .69. Hopkins and Paul [46] made an analysis of thirteen series of junior high school textbooks for a committee of the Department of Superintendence. They list the most common objectives for each grade and give the number of authors who accept each objective. The most common topics of junior high school mathematics are then listed, together with the number of pages devoted to each topic in each textbook.

Algebra textbooks. Overman [73] discusses the need of teaching pupils to use algebraic language and presents an analysis showing the number and type of examples which give practice in algebraic language in each of seven texts intended for ninth grade use. He found that books which are part of a junior high school series are superior from this standpoint to those which are written as algebra textbooks without regard to their relationship to the work of the previous years.

In his extensive study of the learning of algebra, Pease [74] tabulated the number of centimeters of teaching material and the number of drill problems offered in each of three textbooks. There was much variation, caused, as the author says, by the lack of an analysis of the field and of a knowledge of the relative difficulty of the topics. Thorndike [113] discovered that practice on certain algebraic skills in textbooks is distributed in such a manner as to suggest that the authors have made no effort to follow a psychological ideal.

Geometry textbooks. In the field of plane geometry Good and Chipman [41] made an analysis of the topic of proportion in ten textbooks. They report the definitions taught, pages devoted to the topic, the number of exercises, manner of presentation, and the items taught. The texts vary from one-fourth to six in the number of pages devoted to the topic and from five to sixty in the number of exercises used to develop the concept.

Miscellaneous studies of instructional materials. While most studies of instructional materials have concerned themselves with textbooks, Woodring and Sanford [126] list a large number of free and low cost illustrative and supplementary materials which should prove useful to teachers of mathematics. Prices and places where the material is obtainable are indicated in the book.

Many difficulties in the teaching of mathematics can be traced to the pupils' inability to understand the vocabulary used. Remmers and Grant [85] took samples of words from twelve mathematics textbooks, six in algebra and six in geometry, and determined the difficulty of the words from the Thorndike *Teacher's Word Book*. The mathematics textbooks contain from 220 to 380 different words per running thousand, but the most difficult book does not use as many different words as Horace Mann's *Second Reader*. The books have many words which are not in the Thorndike list at all, one as few as twenty-three, and one as many as sixty-two. These

represent the technical vocabulary of the subject. Geometry books vary most, several having a lighter vocabulary load by an amount which is statistically significant. No statistically significant differences were found among algebra textbooks. While the authors did not mention this, an examination of their data shows that when all the books are arranged in order of their difficulty, the five easiest are dated since 1920, while the seven hardest are dated 1915 or earlier. None of the bo ks bear dates between 1915 and 1920.

IX. Testing and Tests

Algebra. Because of the relatively objective nature of the subject matter, algebra lends itself more easily to testing than do some other subjects. This fact has resulted in the construction of a large number of objective tests and the presence on the commercial market of many standardized tests. The construction of a standard test, the virtues of a testing program, and the interpretation of the results of the test are described in some detail by Rugg and Clark [92]. Various commercially available algebra tests are discussed by Ruch and Stoddard [90]. They give grade norms and discuss the reliability and validity of these tests. Correlations between the performance on various tests are also given. The problem of prognosis in algebra is discussed by Orleans and Orleans [72] in a description of the construction of a test which has prognosis for its main purpose. The test described is shown to correlate with achievement to the extent represented by an *r* of .82. Nyberg [70] presents a method for grading non-standardized tests. He shows how examples can be analyzed into their component steps and marked on this basis. He recommends giving part credit, for example, if part of the steps are correct. The purpose is to enable different teachers to grade the same paper uniformly, but he does not cite data to show that this can be done. Harris and Breed [45] made a study of the comparative validity of the Hotz scales and the Rugg-Clark test in algebra. They considered difficulty of items, arrangement of items, convenience, normality of distributions obtained, and diagnostic value, and found that the items in the Hotz test are for the most part arranged in order of difficulty, while the items in the R-C test are not so well arranged. The Hotz test gives fairly regular distributions. The graphing test of the R-C series is declared to be a poor measure since the median is zero. The tests seem to be equally useful for diagnostic purposes.

Several writers have described the construction of standard tests. Douglass [30] asked 100 mathematicians what the "fundamentals" of algebra are. Fifty-nine replied and he selected (1) collection of terms, (2) multiplication, (3) division, and (4) solution of simple equations as topics to be included in a test. Ten items were selected for each section of the test. The tests were given to about 1000 pupils in fourteen schools in five states. and the individual items were given weight according to the percentage of pupils who solved them. Norms were then established. The reliability of the test was later found to be .627. In a later report [31] Douglass describes the construction of Series B of his tests. He asked a number of workers interested in secondary school mathematics what aspects of algebra need to be measured besides those represented in Series A. It was decided to include (1) fractions, (2) factoring, (3) formulas and fractional equations, (4) simultaneous equations, (5) graphs, (6) square root, exponents, and radicals, and (7) quadratic equations. He abandoned weighting because his previous studies showed that the scores obtained on the test without weighting the items correlated closely with the weighted scores. Norms based on the performance of from 315 to 858 pupils in from five to ten schools were established for each group. Monroe [68] describes an algebra test which was given to high school pupils for the purpose of establishing standards. A similar report by Hotz [47] describes the construction of the algebraic scales bearing his name.

Dalman [26] gives a description of the construction and use of a series of instructional tests in algebra intended to facilitate the teacher's task of adapting work to pupils' progress. The classes using the tests made greater progress than those not using them. Three tests were made out on each topic, these tests varying in difficulty. When the teacher thought that about half the class could pass the easiest test, she gave it, and then divided the class for further instruction. Later, those who had passed the first test took the second, while the others repeated the first test. Now she had three groups. Later, the best group took a third test known as the A test, the second group took the B test, while the slow group took the C test a third time. The slow group was given a great deal of individual attention. The author reports that the procedure resulted in improved work.

Some of the more recent writings on the teaching of algebra

have stressed the need of developing functional thinking. This aspect of algebra teaching has not found its way into the standard tests up to the present time. Breslich [12] describes the construction of a test designed to measure functional thinking. A copy of the test, which was given to 1800 pupils, is shown. It has eight parts, measuring eight aspects of the pupil's mastery of the function concept. The author gives the median and range for each semester of the high school. Great individual differences were found, but he concludes that many pupils never acquire the understanding of functionality as illustrated in high school algebra.

Geometry. The pioneer study of the problem of measurement in high school geometry was carried out in 1916 by Stockard and Bell [106]. They worked out a test in geometry involving seventy items. This test was given to six classes, which included 372 pupils. The items were assigned P.E. values according to their difficulties. The construction of a typical geometry test is described by Sanford [94] in a detailed discussion of the Schorling-Sanford geometry test. Correlations for several classes between teachers' judgments and pupils' ranks are given on sections of the test and on the test as a whole. The reliability of the test in various classes ranged from .46 to .85. Minnick [66] constructed tests to measure various aspects of geometric learning. All the tests correlated positively with teachers' marks in all the sixty-three schools in which they were tried, and in most cases the coefficient of correlation was more than three times its probable error. Medians are given for each test. The author believes that the abilities measured can be developed by the time the pupil has completed Book I. He found great variation between schools. In one test the median varied from 38 to 80. In another report [64] the same author describes the construction of a scale designed to measure the ability of pupils to draw construction lines correctly to prove geometry theorems. The difficulty of the items was determined by finding P.E. values based on the results obtained by giving the test to about 700 pupils. Five exercises equally spaced along the line of P.E. values were selected for the scale, which is reproduced in full.

A new type of test is described by Greene and Lane [42]. They constructed a series of six tests to be given at intervals during the school year, each test covering a certain section of the curriculum. Each theorem on the National Committee list was included in the test, and the vocabulary was found to be common to eighteen text-

books. The reliability coefficients for the tests ran from .78 to .90. The various standard tests which are commercially available were studied by Ruch and Stoddard [90]. They describe these tests and give their reliability and validity when obtainable.

Miscellaneous studies of mathematics testing. A comprehensive description of the problems, method of construction, and uses of mathematics tests is given by Reeve [82]. He discusses the place of tests in the teaching program and gives illustrations to show how diagnosis may be made from the results of such tests. A set of scales with items arranged according to a scale unit are included, together with the P.E. values for the items. Stoddard [107] describes the construction of two tests—one, a measure of mathematics training and the other one, a measure of mathematics aptitude. The first was designed to measure the result of the typical mathematics high school course, while the second was an effort to predict the probable success of the individual in college mathematics courses. While these tests were designed for use in colleges as aids to student guidance, they were used to some extent in high schools. Norms are given for college classes and the predictive power of the tests is discussed at some length.

McCoy [61] describes a test in solid geometry which was given to twelve sections in the Boston English High School. A copy of the test, distributions for each section, and a distribution for the test as a whole are included. He discusses the reasons for giving the test and the reactions of pupils. Teachers will find the test useful and interesting, and may compare the results obtained in their own classes with those submitted.

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THE TEACHING OF INTUITIVE GEOMETRY

A STUDY OF ITS AIMS, DEVELOPMENT, CONTENT, AND METHOD

By WILLIAM BETZ

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"Our sole dignity consists in thinking. Let us therefore strive to think correctly. That is the beginning of morality." PASCAL

INTRODUCTORY STATEMENT *

Meaning of intuitive geometry. By "intuitive geometry" we shall denote provisionally that desirable form of elementary geometric instruction which should correspond approximately to the work in arithmetic and in the first principles of algebra, as now provided in our best elementary and junior high schools.

Let it be stated at the outset that the designation "intuitive geometry" is far from satisfactory. Only the fact that thus far a better term has not been evolved or generally accepted suggests its temporary retention. The long list of descriptive adjectives which have been used from time to time to characterize this subject, such as concrete, observational, inventional, mensurational, experimental, constructive, propædeutic, preparatory, empiric, informal, reflects sufficiently the lack of clearness that still prevails on the precise function of this phase of the mathematical curriculum. Each of these alternative designations evidently refers to one particular aspect of content, or method, or aim. Ever since Pestalozzi made

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a pedagogic gospel of *Anschauung* (the German equivalent of "intuition"), there has been a continuous debate in the field of geometry on the real meaning and the implications of this many-hued term. And it is this lack of agreement that appears to be at the bottom of the negative attitude of many high school teachers of mathematics with reference to the claims of intuitive geometry. To them, intuitive geometry only too often has come to mean a hodgepodge of unrelated geometric impressions which the pupil acquires uncritically by a variety of "unscientific" procedures, without rhyme or reason, and devoid of organization, thus causing the worst kind of confusion in the early stages of *demonstrative* geometry.

Needless to say, this type of intuitive geometry, wherever it exists, is a caricature of what the subject *should* be, and it deserves all the criticism that is heaped upon it. Intuitive geometry is neither an inferior type of geometry, nor is it merely a preparation for the "real" geometry of the high school. It has meaning and importance in its own right, being essentially *the geometry of everyday life.*

In the following pages the attempt will be made to offer a constructive analysis of the aims, the development, and the inherent possibilities of this subject, which has been such a storm center for more than a century.

Nature of questions to be discussed. Geometry, as everybody knows, has had a long and honorable career both as a science and as a school subject. Anyone familiar with its background will understand why a necessarily brief monograph like this has to limit itself to the "high spots" of this development. For purposes of a more complete orientation, numerous supplementary references have been added. Unfortunately, many of the most valuable documents relating to the theme under consideration are not available in English translations.

The following list of questions may serve to suggest the general nature and scope of the problems to be discussed:

1. What is the real or alleged educational significance of intuitive geometry?

2. How did intuitive geometry come to be a school subject (*a*) in other countries? (*b*) in America?

3. What are the ultimate sources of geometric knowledge?

4. Is it possible to formulate a compelling type of motivation for the basic content of intuitive geometry?

5. What would seem to be some of the most promising modes of organizing the instructional materials?

6. What suggestions may we obtain from the curricula of other countries?

7. In this field of work, what would seem to be some of the necessary characteristics of an ideal method of teaching?

It is the hope of the writer that the very condensation of the materials offered in these pages may induce others to make more comprehensive studies and thus to assist in throwing more light on the numerous scientific and pedagogic problems that remain to be solved in this field.

PART ONE

The Educational Significance of Intuitive Geometry

A long struggle for recognition. Evidence will be submitted in a later section that for more than seventy years the attempt has been made by American teachers and authors to secure a more adequate provision in the elementary mathematical curriculum for systematic instruction in geometry. And yet, aside from the emphasis put on the fundamental rules of mensuration in the traditional arithmetic course, very little progress was noticeable until quite recently. For a long time, all efforts at improvement seemed unavailing, and were regularly frustrated by considerations such as the following: (1) there is no time for such work; (2) the teachers are not prepared to give such instruction; (3) it would take away the appetite for "real" geometry in the high school.

At last, the arrival of the junior high school movement led to a change, supported and guided by the authority of the National Committee on Mathematical Requirements. Since 1920, a considerable number of textbooks and courses of study have appeared which reflect this new orientation. Even to-day, however, it must be confessed that there is still a considerable diversity of opinion about the precise objectives and the scope of intuitive geometry in our American schools.

The very first step in the direction of remedying the present confusion of objectives would seem to be a clarification of the *aims* of intuitive geometry. We therefore proceed to an exposition of

certain challenging arguments which are intended to provide an adequate educational platform for this branch of mathematics.

I. The Principal Arguments in Favor of Intuitive Geometry

The historical argument. The origin of mathematics carries us back to the dawn of human history. From the beginning, this great subject has been anchored on the bedrock of fundamental human needs. For even the primitive people found it necessary to count and to measure. These two unavoidable activities caused the eventual development of arithmetic and geometry. Thus it is that, from its very inception, mathematics has had a dual foundation and two principal themes. These two themes of mathematics developed side by side throughout the ages. The omission of either one would cripple mathematics beyond repair. Hence, any scheme of mathematical instruction which minimizes or ignores the indispensable rôle of geometry seriously unbalances the curriculum, endangers the pupil's progress, and leads inevitably to mathematical stagnation and inefficiency.

The practical argument. We are living in a world which is incurably mathematical. Number and form accompany us wherever we go. We cannot make or manufacture the simplest object without giving due consideration to its *shape*, its *size*, and the correct *position* of its parts. And as problems of construction become more complex, correspondingly greater demands are made on exact geometric knowledge. Thus, a skyscraper, or an automobile, or a bridge, or a tunnel represents a veritable symphony of applied geometry.

Hundreds of trades depend for their very existence on precise measurement, on blue prints and scale drawings. Maps, charts, and graphs are the very warp of modern travel and commerce, and the calendar we use is made possible only by a continuous survey of the heavens. In short, what subject can boast of a more all-pervasive, practical relation to modern life?

The cultural argument. Many years ago William George Spencer, the father of Herbert Spencer, who wrote an "inventional geometry," penned these interesting lines in support of early instruction in geometry:

When it is considered that by geometry the architect constructs our buildings, the civil engineer our railways; that by a higher kind of geometry,

the surveyor makes a map of a country or of a kingdom; that a geometry still higher is the foundation of the noble science of the astronomer, who by it not only determines the diameter of the globe he lives upon, but as well the sizes of the sun, moon, and planets, and their distances from us and from each other; when it is considered also, that by this higher kind of geometry, with the assistance of a chart and a mariner's compass, the sailor navigates the ocean with success, and thus brings all nations into amicable intercourse . . . it will surely be allowed that its elements should be as accessible as possible.[1]

It has been said that *measurement is the master art.* Let any skeptic ponder the tremendous fact that if number ideas and measurement were suddenly blotted out,

. . . science would be lost, machinery would go out of use, and the civilization of our industrial age would vanish. The mind of man would be weakened through the loss of its keenest weapon. Thought would no longer be precise, or if precision were ever attained, it could not be communicated.[2]

Again, the geometric principles of equality, symmetry, congruence, and similarity are implanted in the very nature of things. Hence every product of the practical or fine arts must reflect these principles. Geometry, even in its simplest aspects, cultivates the ability to visualize, to construct, and to appreciate spatial forms. Creative imagination of this sort is the tool that inspired the masterpieces of all plastic art. Perhaps only the exceptional teacher at present realizes the potential richness of geometry, but these cultural claims of geometry can neither be denied nor ignored.

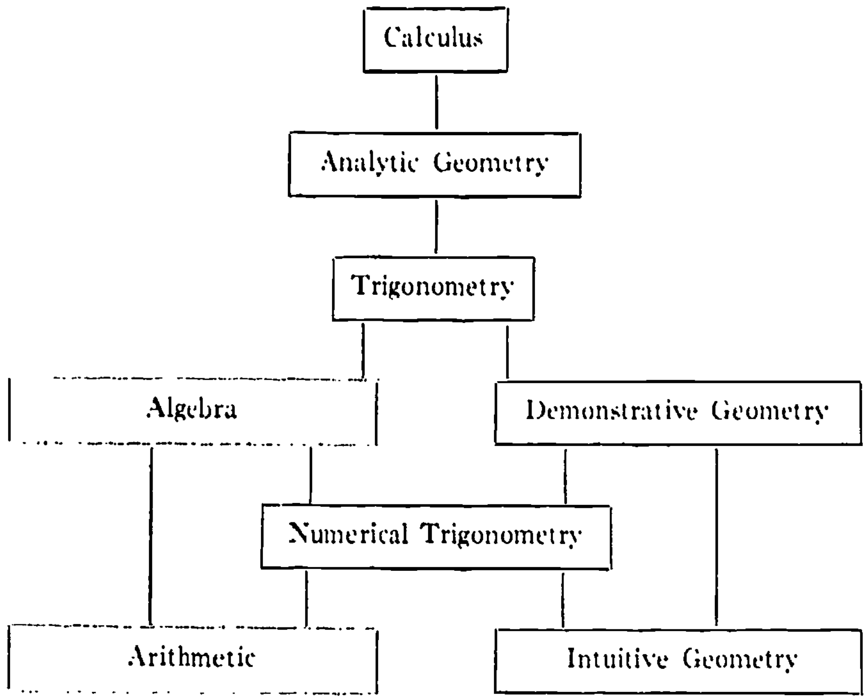
The pedagogic argument. We have seen above that an organic curriculum in mathematics is impossible without intuitive geometry. It deserves an honored place in the course of study because of its own inherent significance, quite aside from the aid it renders to subsequent mathematical work. And yet, its purely *preparatory* value cannot be overestimated. If demonstrative geometry is to

[1] The passage quoted above was part of the introduction which Spencer wrote for the American edition of his book (1876). The complete introduction was reprinted by Reeve, W. D., in *The Fifth Yearbook, The National Council of Teachers of Mathematics*, 1930, pp. 8-9. The original English edition came out about 1830 or 1835 (according to Cajori, Florian, in the *Final Report of the National Committee of Fifteen on Geometry Syllabus*, reprinted in *The Mathematics Teacher*, Vol. V, No. 2, pp 41-131, December, 1912.).

[2] Quoted from an address by Buckingham, B. R., on "Statistics and Modern Educational Thought." *School and Society*, March 22, 1930, p. 380.

survive as a school subject, it must have a real foundation comparable to that which arithmetic furnishes for the study of algebra. At present too many pupils flounder in demonstrative geometry

ELEMENTARY MATHEMATICS



because of the almost simultaneous appearance of too many difficulties. Within a few weeks, the perplexed beginner is expected (1) to master a formidable list of concepts and definitions, (2) to acquire skill in handling geometric instruments, (3) to face in a totally new domain the niceties of a logical procedure which nothing has prepared him to appreciate. Many of these difficulties would vanish, or would be greatly reduced, if we had a continuous six-year course in secondary mathematics in which all the work were properly correlated, and having as its foundation stones arithmetic and intuitive geometry, as suggested in the diagram above.[3]

The psychological argument. The cultivation of space intuition and of plastic thinking is characteristic of all properly con-

[3] See *A Tentative Syllabus in Junior High School Mathematics*, p. 7, issued by the University of the State of New York, new edition, Albany, 1931.

ducted work in geometry. The significance of this type of training as a basic feature of *all* mental work has been brought to light in an almost spectacular manner by the most fruitful of recent psychological developments. The "configuration" or *Gestalt* point of view in psychology states that our impressions of the outer world come to us in organized "wholes," or in "patterns," the parts of which, in each case, are only appreciated in relation to the whole. This doctrine has been set forth recently in great detail by Professor Raymond H. Wheeler of Kansas University.[1] The relations of this psychological theory to mathematics, and to geometry in particular, has been explained in a most fascinating way by Professor Gardner Murphy of Columbia University. In an article entitled "The Geometry of Mind," he says:

> But why, you ask, must the world be *patterned* at all? Why can there not be things existing for their own sweet sake, detached, serene, and unorganized? *Why must things always be related to other things and the total patterns thus constituted be related in turn to still other patterns?* Why, in fact, when mathematics is such a burden to nearly all of us, *should the structure of the world be mathematical at all?* Well, on this point you will have to consult the architect of the universe. Just as the Abbé Haüy found that crystals formed in some ways and not in others, because numbers were so and so; just as Mendelyeev found that the elements arranged themselves periodically because quantity was of the very nature of the supposed qualitative difference between chemical elements; just as studies of heredity have found that the almost infinitely complicated characteristics of the different members of a species are due not to mere likeness or unlikeness to grandparents, but to the permutations and combinations of independent "genes" behaving according to the law of probability, so, even the psychologist, he who presides over the inner recesses of the mind, finds that it is a stranger, the *mathematician*, and not he himself, *who has the key to the treasure.*[2]

In reading these and many similar statements, who is not reminded of Plato's famous dictum, "God always geometrizes"?

We have thus seen that there are excellent *theoretical* grounds for the persistent attempts that have been made in recent years to give a more prominent place to systematic geometric instruction in the curricula of our American schools.

[1] See Wheeler, R. H., *The Science of Psychology*, Thomas Y. Crowell Company, New York, 1929; Wheeler, R. H. and Perkins, F. T., *Principles of Mental Development*, Thomas Y. Crowell Company, New York, 1932; Wheeler, R. H., *The Laws of Human Nature*, D. Appleton and Company, New York, 1932.

[2] Murphy, Gardner, "The Geometry of Mind," *Harper's Magazine*, October, 1931, p. 592.

II. Evidence and Opinions in Corroboration of These Claims

A. The Experience of Other Countries

Reports of the International Commission on the Teaching of Mathematics. Many of the leading countries of the world have come to a substantial agreement in their attitude toward the claims of intuitive geometry. A survey setting forth its status in fifteen foreign countries is included in Mr. J. C. Brown's summary of the world's mathematical curricula.[6] *The Fourth Yearbook, The National Council of Teachers of Mathematics* (1929) presents a more up-to-date supplementary picture of the situation in twelve foreign countries, including Russia and Japan.[7] Other material on the subject is furnished by recent courses of study and by the current textbooks used in certain of these countries.

It has been found that intuitive geometry, when properly presented, vitalizes and unifies the whole course in elementary mathematics. Certain phases of intuitive geometry are not only of greater interest to children than are the stereotyped applications of business arithmetic, but they are also of greater immediate importance mathematically. These conclusions, an outgrowth of over a century of experimentation, are in complete harmony with the views of an ever-increasing group of American educators, as the following testimonies will prove.

B. Appraisal of Intuitive Geometry by American Educators

Early endorsements of the subject. Perhaps the first noteworthy attempt to create a wider demand for a first course in geometry on the part of American teachers was due to Thomas Hill, a New England clergyman and one-time president of Harvard College. In 1854 he completed a fascinating little book of 144 pages which he called *First Lessons in Geometry*. The motto on the title page reads, "Facts Before Reasoning." The following quotation from the preface is of singular interest even to-day:

[6] Brown, J. C., *Curricula in Mathematics*, United States Bureau of Education, Bulletin, 1914, No. 45. This document in condensed form constitutes Chap. XI of *The Reorganization of Mathematics in Secondary Education*, a report issued by the National Committee on Mathematical Requirements under the auspices of the Mathematical Association of America, Inc., 1923.

[7] This volume is entitled *Significant Changes and Trends in the Teaching of Mathematics Throughout the World Since 1910*.

I have long been seeking a Geometry for beginners, suited to my taste, and to my convictions of what is a proper foundation for scientific education. Finding that Mr. Josiah Holbrook agreed most cordially with me in my estimate of this study, I had hoped that his treatise would satisfy me, but, although the best I had seen, it did not satisfy my needs. Meanwhile, my own children were in most urgent need of a textbook, and the sense of their want has driven me to take the time necessary for writing these pages. *Two children, one of five, the other of seven and a half, were before my mind's eye all the time of my writing;* and *it will be found that children of this age are quicker of comprehending first lessons in Geometry than those of fifteen.*

Many parts of this book will, however, be found adapted, not only to children, but to pupils of adult age. The truths are sublime. I have tried to present them in simple and attractive dress.

I have addressed the child's imagination, rather than his reason, because I wished to teach him to conceive of forms. The child's powers of sensation are developed, before his powers of conception, and these before his reasoning powers. This is, therefore, the true order of education; and *a powerful logical drill*, like Colburn's admirable first lessons in Arithmetic, *is sadly out of place in the hands of a child whose powers of observation and conception have, as yet, received no training whatever.* I have, therefore, avoided reasoning, and simply given interesting geometrical facts, fitted, I hope, to arouse a child to the observation of phenomena, and to the perception of forms as real entities.[5]

A few years later, Thomas Hill issued his *Second Book in Geometry* (1863), not comparable with the first in merit, using as the motto on the title page, "Reasoning Upon Facts." In the preface he states that the first book was intended for children from six to twelve years of age, while the second book was adapted for the use of "children" from thirteen to eighteen years old.

A second pioneer effort on behalf of intuitive geometry is represented by a text also entitled *First Lessons in Geometry*, written by Bernhard Marks, principal of the Lincoln School in San Francisco. It was published in New York in 1871. The title page states that it was "designed for the use of primary classes in grammar schools, academies, etc." In the preface the author refers approvingly to the work of "President Hill of Harvard College." The following passages from the preface are of particular significance:

How it ever came to pass that Arithmetic should be taught to the extent attained in the grammar schools of the civilized world, while Geometry is

[5] Quoted by Reeve, W. D., in *The Fifth Yearbook, The National Council of Teachers of Mathematics*, 1930, p. 10.

almost wholly excluded from them is a problem for which the author of this little book has often sought a solution, but with only this result; viz., that Arithmetic, being considered an elementary branch, is included in all systems of elementary instruction; but Geometry, being regarded as a higher branch, is reserved for systems of advanced education, and is, on that account, reached by but very few of the many who need it.

The error here is fundamental. Instead of teaching the elements of all branches, we teach elementary branches much too exhaustively.

The elements of Geometry are much easier to learn, and are of more value when learned, than advanced Arithmetic; and, if a boy is to leave school with merely a grammar-school education, he would be better prepared for the active duties of life with a little Arithmetic and some Geometry, than with more Arithmetic and no Geometry.

Thousands of boys are allowed to leave school at the age of fourteen or sixteen years, and are sent into the carpenter-shop, the machine-shop, the mill-wright's, or the surveyor's office, stuffed to repletion with Interest and Discount, but so utterly ignorant of the merest elements of Geometry, that they could not find the centre of a circle already described, if their lives depended upon it.

Unthinking persons frequently assert that young children are incapable of reasoning, and that the truths of Geometry are too abstract in their nature to be apprehended by them.

To these objections, it may be answered, that . . . nothing can be more palpable to the mind of a child than forms, magnitudes, and directions. . . . The author holds that this science should be taught in all primary and grammar schools, for the same reasons that apply to all other branches.

If this view is correct, *it is wrong to allow a pupil to reach the age of fourteen years without knowing even the alphabet of Geometry. He should be taught at least how to read it.*

It certainly does seem probable, that if the youth who now leave school with so much Arithmetic, and no Geometry, were taught the first rudiments of the science, thousands of them would be led to the study of the higher mathematics in their mature years, by reason of those attractions of Geometry which Arithmetic does not possess.[9]

For a considerable period of years New England appears to have been the most active center of experimentation in this field of work. About ten years after Marks's book appeared, Mr. G. A. Hill of Cambridge, Massachusetts, wrote an admirable *Geometry for Beginners*[10] which reflected a greatly increased interest in the subject.

By 1893, the grammar schools of Cambridge had begun to include the elements of intuitive geometry in their curriculum. In that year, Professor Paul H. Hanus of Harvard University gave

[9] Marks, Bernhard. *First Lessons in Geometry*, pp. [illegible]. Ivison, Blakeman, Taylor and Co., New York, 1871.

[10] Hill, G. A., *Geometry for Beginners*, Ginn and Company, Boston, 1880.

a course of lectures "to the teachers of the seventh, eighth, and ninth grades." These lectures "formed part of the plan whereby Harvard University gave instruction to teachers of the grammar schools in certain new subjects introduced into the curriculum." From these lectures there evolved a famous "essay," entitled *Geometry in the Grammar School*. The following quotations from the first two pages of this pamphlet, while still revealing the influence of the old "faculty psychology," now abandoned, may serve to amplify the practical and the cultural arguments outlined above.

Like every other subject admitted into the curriculum, geometry has a value as knowledge and a value as discipline. Its value as knowledge is illustrated in all the manifold occupations of life; *most of the trades and industries have a mathematical basis.* Every worker in wood, tin, iron, stone —whether contractor, builder, tinsmith, mason, carpenter, or what not— must appeal to geometry to avoid waste of material and loss of time in useless experimenting; so also the farmer who wishes to know the extent of a piece of land in order to estimate the cost of reclaiming it or the amount of its possible productiveness, who has a drain or ditch to build, or a dam to construct, keenly feels the lack of such an elementary knowledge of geometry as would enable him to make simple preliminary measurements and save him both time and money.

Thus *the knowledge value of geometry is obviously very great.* Turn now to its disciplinary value. In its early lessons geometry develops correct notions of form and magnitude, and an appreciation of regularity and proportion; and through the definiteness and precision of its data it trains the mind to accuracy and clearness in grasping conditions. The way is thus easily and naturally prepared for a strictly scientific study of the subject later on, when through rigorous procedure from step to step by reference to fundamental principles, or previously established propositions, and the ruthlessness with which every error is detected and overthrown, it leads to the consciousness of full and solid achievement—the most powerful of all intellectual incentives to exertion. A firm grasp upon data, a rigorous exactness in reasoning, and the consequent achievement help to form the habit of insisting upon nothing less than adequate mastery, whatever the subject may be. This habit marks the trained mind.

We conclude, therefore, that *geometry has a place in the curriculum of the grammar school because it yields a peculiar and important kind of knowledge and a highly desirable mental discipline.*[13]

During this earlier period the most influential appraisal was that of the famous Committee of Ten (1894).[14] The membership

[13] Hanus, Paul H., *Geometry in the Grammar School*, pp. 1-2, D. C. Heath and Company, Boston, 1894.

[14] See the *Report of the Committee of Ten on Secondary School Studies*, pp. 104-116, published for the National Educational Association by the American Book Company, Inc., New York, 1894.

of its mathematical subcommittee included such outstanding names as these: Professor Simon Newcomb of Johns Hopkins University, the distinguished astronomer; Professor Florian Cajori, the learned mathematical historian; Professor Henry B. Fine, late dean of Princeton University; and the honored and beloved Professor George D. Olds, late president of Amherst College. The committee's "Special Report on Concrete Geometry" is reproduced in full on page 97 of this *Yearbook*.

More recent opinions of leading educators. Only a few typical quotations of more recent date can be included at this point. They extend over a period of several decades, but have lost nothing of their actuality on that account.

Professor G. Stanley Hall, the distinguished educationist of Clark University, in one of his scholarly treatises commented as follows on the place of intuitive geometry in the mathematical curriculum:

Elementary geometry should come very early, but in the form of manifold curious and enticing appeals to the eye, for optical geometry is a very potent agent in arousing curiosity and interest, and "preforms" the demonstration of geometry, which is to come later. It was such things that originally evoked the process of geometrizing and the buds and rudiments of this power are big in the child. . . . While this intuitional type of geometry should *come earlier than now, almost at the beginning of arithmetic*, algebraic elements should come much later than pedagogues or mathematicians now advise, for they are hard because abstract.[13]

The *Thirteenth Yearbook of the National Society for the Study of Education* contains a monograph on "Reconstructed Mathematics in the High School," written by Professor Henry C. Morrison, now of the University of Chicago. We find in it these interesting passages concerning the place of intuitive geometry in the curriculum:

There is probably little or nothing in the way of introducing the type of geometrical study which I have described at any time after about the twelfth year, but the *earlier the better*.

The processes taught must be *applied mathematics* brought into the program *at the point* where *it will be used and taught as a body of principles directed to a known and felt need and not as a logical system.* The order of introduction should probably be: *geometry and a great deal of it, then the necessary algebra*, and finally so much of the *higher mathematics as will suffice*.

[13] Hall, G. Stanley. *Educational Problems*, Vol. II, pp. 395-396 D. Appleton and Company, New York, 1911.

The traditional round of mathematics in the high school, to wit: elementary algebra, plane and solid geometry, trigonometry, and advanced algebra, must be revised both as to organization and content, and adapted to the known nature of the adolescent and to the social purpose of the high school as that purpose is increasingly revealed by modern conditions.

Mathematics must be treated primarily as a *language*, the purpose of which is the interpretation of the various sciences.

Courses in mathematics must be arranged at such points in the curriculum as will give immediate opportunity for functioning.[14]

In an article that appeared in the May, 1918, number of *Education*, President Charles W. Eliot made a plea for an earlier introduction of geometry. He said:

The amount of time claimed for arithmetic hurts seriously the whole course of study. . . . Instruction in geometry has been begun too late.

Professor Charles H. Judd, of the University of Chicago, in his *Psychology of High School Subjects*, states in bold relief some crucial arguments in favor of intuitive geometry. To quote:

The student who knows the abstract demonstrations of geometry, but does not realize that knowledge of space is involved in every manufacturing operation, in every adjustment of agriculture and practical mechanics is only half trained. . . . The direct perceptual experience which is most closely related to all types of mathematical thought is *space*. Space, because of its character as a relational type of experience, is not only itself a natural subject of mathematical consideration, but is also capable of representing in graphic form those mathematical relations which are usually represented in letters and numbers. Space is therefore strongly suggested as an instrument for both the exemplification and the expression of mathematical ideas. Furthermore, by virtue of the intimate relation of space perception to mechanics, *space seems to be a good instrument for the training of students in application of mathematics.* While thus emphasizing the significance of space for mathematics, it is proper once more to emphasize the historical fact that *in our Western civilization the science of space is prior to all other phases of mathematics. It is altogether probable that this fact will shortly be recognized in the elementary course.* . . .

If this book were intended for teachers in the elementary schools, it would advocate a course in form study early in the grades, and it would advocate the use below the high school of some of the economical methods of mathematical reasoning taught by algebra. . . .

. . . Teachers in the lower schools have never realized that the union of logic and space studies deprived them of one of their most natural subjects of

[14] Morrison, Henry C., "Reconstructed Mathematics in the High School," *The Thirteenth Yearbook of the National Society for the Study of Education*, pp. 9-32, University of Chicago Press, Chicago, Ill., 1914.

instruction, namely, *form-study. The logical statement of the principles of geometry has blinded modern as well as medieval teachers to the true worth of this subject for younger pupils.*[15]

PART TWO

The Development of Intuitive Geometry as a School Subject

Difficulty of this investigation. A complete presentation of all the factors which influenced the development of intuitive geometry as a school subject would have to take account of (1) the historic evolution of geometry as a science, and (2) the gradual growth of the elementary and secondary curricula in all the leading countries. For obvious reasons, our program must be a very much more modest one. The period of development in which we are primarily interested dates approximately from the time of Pestalozzi, and is coextensive with the creation of common schools "for all the children of all the people." Even with this limitation, however, it would be necessary to summarize a vast body of literature far exceeding the permissible scope of this monograph, as well as the available resources of any individual writer. Hence, we shall be obliged to select only those aspects of the story of intuitive geometry which are likely to be of interest and value in connection with present educational tendencies and problems.

Modes of approach. First of all, it should be kept in mind that by the time geometry became a subject of instruction in the modern school, all its fundamental facts and principles had become known, most of them in the days of the ancient Greeks. Unlike the natural sciences, which have been in a violent state of flux since the days of Kepler, Galileo, and Newton, *geometry has had a practically fixed content.* In the nineteenth century, it is true, revolutionary changes took place in our conception of the *foundations*, but this tremendous upheaval was of too philosophic a character to affect to a marked extent the point of view of the first course in geometry. Hence the teachers of elementary geometry at all times were primarily concerned with problems of *selection* and of *method.* Lines, angles, circles, triangles, polygons, the geometric solids, considerations of symmetry, congruence, similarity, and the technique of mensuration—these were the basic materials that formed the

[15] Judd, Charles H., *Psychology of High School Subjects*, pp. 21, 130-132. Ginn and Company, Boston, 1915.

framework of each geometric course of study, in every imaginable transformation and combination. The *concepts* and *facts* did not have to be created anew, but each generation of teachers felt at liberty to try a different point of departure and to experiment with ever-changing classroom procedures. And it has remained so to this day. Many and varied are the approaches to the great and eternal landmarks of geometry. Essentially, however, all these roads fall into one of two main groups. There are those who prefer the slow pedestrian route of *systematic* exploration, gradually piecing together into an ever-growing mosaic the daily acquisitions of new concepts, skills, facts, relations, and applications. But there are others who reject this *synthetic* approach. They desire to obtain at once a larger orientation, a panoramic, three-dimensional view of the landscape, and then, by a process of *analysis* and differentiation, to secure an understanding of the constituent parts.

A fascinating spectacle is presented by the manner in which the national temperament of various countries responds to either of these routes. In general, the Germanic countries have shown a tendency to emphasize a gradual and systematic type of development. The Romance countries reveal a passion for clearness and logical coherence, even in the elementary stages. In America, however, the desire for "activities" and immediate practical application has tended to overshadow considerations of thoroughness and scholarship. In our Western civilization the question always seems to be, "What can I *do* with this new type of knowledge or training?" The problem of the future would seem to be that of effecting a compromise between the extreme European and American tendencies.

An important distinction. The school systems of Europe still have a dualistic organization. That is, the elementary school subjects are taught not only in the common schools, which correspond to our public "grammar schools," but also in the lower grades of the secondary schools. The point of view of the common elementary schools is that of *immediate life preparation*, and hence of simple and practical objectives. In the secondary schools, however, the primary aim is *scholarship and preparation for higher institutions of learning*. Hence, when examining European textbooks and courses of study, one must always keep in mind whether the material under examination is intended for the "masses" or the "classes."

In contrast to European practices, it has been the well-nigh in-

superable problem of education in the United States to develop a *unitary system* of instruction that shall be of benefit to every learner without exception.

I. The Fateful Influence of Euclid

The cult of Euclid. Geometry had the fortune, but also the misfortune, of a masterful compilation, early in its career, in the form of Euclid's *Elements*. Henceforth, for more than two thousand years, Euclid dominated the geometric scene to such an extent that it became almost a sacrilege to deviate in any way from his sacred text. Why did this treatise have such an amazing influence, comparable only to that of the Bible itself?[16] It is, as Professor C. J. Keyser beautifully expresses it, the most famous example of autonomous thinking, of the *postulational method*, in the whole history of science. He says:

> When it was produced, it was so incomparably superior to any other product of human thinking that men were dazzled by it, blinded by its very brilliance, so much so that, though they admired it and in a sense understood it, they failed to perceive that its chief significance was, not *geometrical*, but *methodological*.[17]

To this day it has been virtually impossible to break this fatal misconception which has ruined the mathematical education of generations of pupils, and which is mainly responsible for the negative reputation of geometry as a school subject. Only in recent decades has the light of a clearer understanding begun to affect the first steps of a beginner in geometry, long after the "cult" of Euclid had been rendered untenable by mathematicians of unquestioned authority.

In 1849 De Morgan wrote as follows:

> If the study of Euclid has been almost abandoned on the continent, and has declined in England, it is because his more ardent admirers have insisted in *regarding the accidents of his position as laws of the science*.[18]

[16] See Smith, D. E., "Euclid," *The Teaching of Geometry*, Chap. V, Ginn and Company, Boston, 1911. On p. 47 Smith states that Riccardi in 1887 listed "well towards two thousand editions" of Euclid. The standard English edition is that of Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements*, in 3 vols., Cambridge, 1908; second edition, 1926.

[17] Keyser, C. J., *Thinking About Thinking*, p. 26, E. P. Dutton and Co., Inc., New York, 1926.

[18] Quoted by Cajori, Florian, in the *Final Report of the National Committee of Fifteen on Geometry Syllabus*, p. 25, *op. cit.* [1].

Twenty years later, at a meeting of the British Association, J. J. Sylvester expressed his views on the same subject in this manner:

> I should rejoice to see mathematics taught with that life and animation which the presence and example of her young and buoyant sister (natural science) could not fail to impart. short roads preferred to long ones. *Euclid honorably shelved, or buried "deeper than e'er plummet sounded" out of the school boy's reach.*[19]

In spite of these strong protests, Euclid's influence continued almost unabated in England until a veritable explosion occurred in the form of the Perry Movement (1901), which demanded a complete divorce from Euclid.

The revolt against Euclid. The precise details of the *modern* career of Euclid, ever since the appearance of the first Latin translation from Arabic sources, in the twelfth century, may be read in the standard histories of mathematics.[20] We are here concerned principally with the *pedagogic shortcomings* of the *Elements*, since it is primarily these which, throughout the centuries, delayed the development of more sane methods of geometric instruction. Among these defects the following are the most outstanding:

1. Euclid's *Elements* represents a severely logical, *deductive* array of propositions. The proofs are given in *synthetic* rather than *analytic* form. *The method by which the proofs were discovered is completely concealed.*
2. The *sequence* of the propositions cannot be understood or appreciated until the end of the journey has been reached.
3. *Practical applications are entirely ignored.*
4. The study of *solids*, and hence the use of *three-dimensional thinking*, is postponed to the end of the course.
5. The mode of presentation is pedantic and dry-as-dust.
6. No provision is made for a psychological development of geometric concepts, skills, and types of thinking.[21]

[19] *Ibid.*, pp. 25-26.

[20] See Smith, D. E., *op. cit.* [16], Chaps. III, IV, and VI; Heath, Thomas L., *op. cit.* [10]; Stamper, Alva W., *A History of the Teaching of Elementary Geometry*, pp. 27 ff., Bureau of Publications, Teachers College, Columbia University, New York, 1909; Smith, D. E., *History of Mathematics*, Vol. I, Chap. IV, Ginn and Company, Boston, 1923; Sanford, Vera, *A Short History of Mathematics*, pp. 208-275, Houghton Mifflin Company, Boston, 1930. [All bracketed numbers in footnotes are references to the previous footnote in which full information is given about the item listed.—*Editor.*]

[21] See Stamper, Alva W., *op. cit.* [20], Chap. II; Cajori, Florian, in *op. cit.* [18], pp. 5-32; Enriques, F., *Questioni riguardanti la geometria elementare*, or its German translation, *Fragen der Elementargeometrie*, Vol. I, pp. 24 ff., second edition, Leipzig, 1923; Branford, B., *A Study of Mathematical Education*, pp. 240-242, 330-345, Clarendon Press, Oxford, 1924; Klein, F., *Elementar Mathematik*

In other words, Euclid's work was not meant to be a school subject, but rather a "philosophic treatise intended for mature minds."

Sooner or later the universal revolt against Euclid was bound to crystallize into concrete suggestions for improvement, and into experimental attempts to create a new edifice for geometry. In America the need of a new orientation has long been recognized. Perhaps the most authoritative exposition of the newer point of view is that which came from the able pen of the late Professor J. W. Young. The following quotations are taken from two chapters of his *Fundamental Concepts of Algebra and Geometry:*

The chief end of mathematical study must be to make the pupil *think.* If mathematical teaching fails to do this, it fails altogether. The *mere memorizing of a demonstration in geometry has about the same educational value as the memorizing of a page from the city directory.* And yet *it must be admitted that a very large number of our pupils do study mathematics in just this way.* There can be no doubt that the fault lies with the teaching. . . . Mathematical instruction, in this as well as in other countries, is laboring under a burden of century-old tradition. Especially is this so with reference to the teaching of geometry. . . . Our texts in this subject are *still patterned more or less closely after the model of Euclid,* who wrote over two thousand years ago, and whose text, moreover, was *not intended for the use of boys and girls, but for mature men.*

The trouble in brief is that the authors of practically all of our current textbooks lay all the emphasis on the formal logical side, to the almost complete exclusion of the *psychological, which latter is without doubt far more important at the beginning of a first course in algebra or geometry.* They fail to recognize the fact that the pupil has reasoned, and reasoned accurately, on a variety of subjects before he takes up the subject of mathematics, though this reasoning has not perhaps been formal. In order to induce a pupil to think about geometry, it is necessary *first to arouse his interest and then to let him think about the subject in his own way.* This first and difficult step once taken, it should be a comparatively easy matter gradually to mold his method of reasoning into a more formal type. The textbook which takes due account of this *psychological* element is apparently still unwritten, and as the teacher is to a large extent governed by the text he uses, the failure of mathematical teaching is not altogether the fault of the teacher.

Let the teacher once fully realize that his science, even in its most elementary portions, is alive and growing, let him take note of the manifold changes in point of view and the new and unexpected relations which these changes disclose, let him further take an active interest in the new develop-

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vom höheren Standpunkte aus, Vol. II, pp. 203-262, third edition, Berlin, 1925 (the most searching modern critique of Euclid); Fladt, K., *Euklid,* Berlin, 1927.

ments, and indeed react independently on the conceptions involved—for *an enormous amount of work still remains to be done in adapting the results of these developments to the requirements of elementary instruction*—let him do these things, and he will bring to his daily teaching a new enthusiasm which will greatly enhance the pleasure of his labors and prove an inspiration to his pupils.

The teacher's problem is, especially at the beginning, far more psychological than logical. It is obvious that a subject must be presented to a boy of fourteen years in a different way from that employed in presenting it to a mature mind. In particular, it is necessary, in the beginning, to *make continued and insistent appeal to concrete geometric intuition.* From this point of view, the notions of point, straight line, and plane may be assumed to be sufficiently clear in the pupil's mind without any formal definition. Moreover, it hardly seems necessary to say that *many assumptions which are essential in a purely formal logical development of the science may and should be tacitly assumed in a first course.* This is merely another form of the assertion that the power of abstraction and the amount of formal reasoning expected of a pupil at a given time must be *adapted to his capacity to form such abstractions and formal deductions at that time. His capacity in this direction will slowly but surely increase, if it is allowed to develop naturally; it will be greatly impaired, if not altogether destroyed, by any attempt to force its growth.*

The best way to take account of this psychological element would seem to be *the removal of all formal considerations from the beginning of the course in geometry: as much as a half of the first course might profitably be devoted to an informal treatment of geometry*, in which the pupil is made familiar with the more important figures and constructions, and in which he is encouraged to think about the problems which present themselves *in his own way*. During this part of the course the pupil could be led to see the advantages of the more formal methods that follow. Unfortunately none of our present textbooks provide for such an informal introduction.[22]

We shall now try to obtain at least a glimpse of the manner in which some of the leading countries reacted to the necessity of rebuilding their first courses in geometry. Following the admirable suggestion of Professor Young, we shall investigate in particular the *psychological* transformation of elementary mathematical instruction. This leads us at once to an examination of the controlling influence exerted by Pestalozzi, Herbart, and Fröbel. While the ideas of these great reformers now seem rather crude, they prepared the way for the profound changes to follow.

[22] Young, J. W., *Fundamental Concepts of Algebra and Geometry*, pp. 4-7, 104 105. The Macmillan Company, New York, 1911. Many other passages in this volume will be found equally trenchant.

II. INTUITIVE GEOMETRY IN THE GERMANIC COUNTRIES (AUSTRIA, GERMANY, SWITZERLAND)

A. THE ELEMENTARY SCHOOLS [23]

A new spirit. Up to the time of Pestalozzi (1746-1827) elementary geometric instruction had been limited essentially to a few practical rules of mensuration. Its general educational value was not recognized. It required the unexcelled educational genius of a Pestalozzi to take the first real step in the direction of "psychologizing education." This step marked a turning point in the history of modern education, for it placed the emphasis not merely on the subject matter to be taught, but on the developing *mind* of the child and on the manner in which the *learning process* should be carried on. That is, the *method of instruction* and the *spirit* of the schoolroom became of paramount importance. Once this step was taken, all elementary teaching had to be revolutionized. We are still in the very midst of the great transformation inaugurated by the Pestalozzian movement.

The central rôle of intuition. The foundation stone of Pestalozzi's method, the very center of his educational program, was an absolute insistence on *Anschauung*. *He made real experience with natural objects the fundamental starting point of instruction.* He rebelled against the meaningless *verbalism* of the schools, the practice of teaching children words and phrases they could not appreciate. At all times he wished to have the pupil understand the *object before the word,* the *idea rather than the form.*

Kant had distinguished carefully between "empiric" and "pure"

[23] The discussion presented in the following section is based chiefly on the sources below:
(1) Treutlein, P., *Der geometrische Anschauungsunterricht*, Leipzig, 1911.
(2) Timerding, H. E., *Die Erziehung der Anschauung*, Leipzig, 1912.
(3) Höfler, Alois, *Didaktik des mathematischen Unterrichts*, Leipzig, 1910.
(4) Lietzmann, W., *Methodik des mathematischen Unterrichts*, Vol. II, pp. 65-82, Leipzig, 1923.
(5) Engel, Ernst, *Raumlehre*, Langensalza, third edition, 1920.
(6) Büttner, A., edited by Teichmann, O., *Anleitung für den Rechen- und Raumlehre Unterricht*, Neuzeitliche Raumlehre, Vol. V, twenty fifth edition, Leipzig, 1930.

The Austrian territory is included because of the close educational affiliations of Germany and Austria. The earlier Austrian curricula and textbooks, in intuitive geometry, surpassed and anticipated those of Germany by several decades. See Höfler and Treutlein.

intuition. By "empiric" intuition he meant the "direct perception of actual objects."[24] This conception coincides with the usual, everyday interpretation of intuition. "Pure" intuition, on the other hand, according to Kant, is something very different. It is that which *we ourselves contribute* during an act of perception, by virtue of the very structure of our senses and of our mind. Hence Kant regarded the results obtained by "pure" intuition as absolutely certain. Concrete observation was merely the external stimulus which brought into play the activities of "pure" intuition. "Concepts without percepts are empty; percepts without concepts are blind," became one of Kant's most famous sayings. How very similar to Comenius' equally famous motto: *Quod non fuerit prius-quam in sensu, non erit in intellectu!*

It is more than likely that Pestalozzi's view of *Anschauung* was derived directly from Kant. At any rate, it is certain that he also used the word in this same dual sense, a fact which must be kept in mind in a perusal of the extensive subsequent literature dealing with intuition.

In his celebrated *Gertrude*, Pestalozzi characterized the central importance of intuition as follows:

> If I look back and ask myself what I have really done toward the improvement of the method of elementary instruction, I find that in recognizing *Anschauung* as the absolute basis of all knowledge, I have established the first and most important principle of instruction. Moreover, I have tried to discover the essence of this principle, as well as the original method by which Nature herself operates in the educational development of human beings.[25]

Pestalozzi's influence.[26] It is only too true that Pestalozzi in reality brought to a culmination ideas, principles, and tendencies which had been foreshadowed by others, notably by Comenius and Rousseau. Much of his own work was crude, erroneous, and even eccentric. It was Comenius who exclaimed: *Non verba, sed res.* The *Orbis Pictus* of Comenius testifies to his realization of the importance of *Anschauung*.[27] The educational realists of the seven-

[24] Kant, Immanuel, *Prolegomena zu einer jeden künftigen Metaphysik*, p. 54, Riga, 1783.

[25] See Timerding, H. E., *op. cit.* [23], p. 9; also Parker, S. C., *The History of Modern Elementary Education*, p. 324, Ginn and Company, Boston, 1912.

[26] For an excellent account of the Pestalozzian movement, see Parker, S. C., *op. cit.* [25], Chaps. XIII-XVI.

[27] The *Orbis Pictus* (*The World in Pictures*) was published in 1658. It was for many years the most widely used Latin textbook, and editions of it were

teenth and eighteenth centuries, guided by thinkers like Rabelais, Montaigne, Bacon, and Ratke—the influence of John Locke and the association psychology, as well as the writings of the "philanthropists"—had created a background that anticipated the essentials of Pestalozzi's doctrines. To Rousseau's *Émile* Pestalozzi himself repeatedly acknowledged his great indebtedness.[28]

Pestalozzi's main contribution was his unparalleled and untiring devotion to the new ideas, his passionate enthusiasm on their behalf, and the missionary spirit with which he inspired those who followed him. Among his most ardent admirers and followers were outstanding men like Herbart and Fröbel, each of whom had made a careful study of the Pestalozzian methods in Pestalozzi's own institutions.

These disciples of the master were destined to affect profoundly the educational progress of both European and American schools. S. C. Parker goes so far as to assert that few modern school practices are not Pestalozzian or Herbartian. He says:

> When we have selected from recent educational practices those that are Pestalozzian and those that are Herbartian either in origin or character, there remain very few aspects of the modern elementary school to be discussed.[29]

Form study, a basic feature of the movement led by Pestalozzi. Like Kant, Pestalozzi saw in mathematics the most noteworthy expression of "pure" intuition. He came to regard language, number, and form as the basic elements of all education. As to "form," if one views the external shapes of physical objects, if one tries to understand and to reproduce the constantly recurring elements of these shapes, one is led inevitably to the type of geometry which has always been of primary interest to the artist, the engineer, and the craftsman. And so, in his pioneer days at Burgdorf, Switzerland, Pestalozzi for a time subordinated all other considerations to a cultivation of geometric ideas, and to related work in geometric drawing. Even penmanship, reading, and arithmetic he considered to be definitely based on geometry, or the art of *mensuration*, as he called it.

issued as late as the nineteenth century. The eleventh English Latin edition appeared in London in 1728. See Parker, S. C., *op. cit.* [25], pp. 140 ff.; also Monroe, P., *Textbook in the History of Education*, pp. 492 ff., The Macmillan Company, New York, 1918.

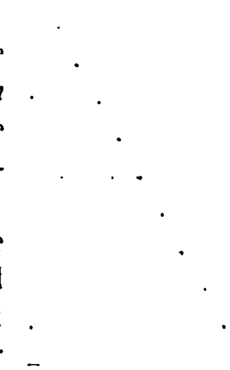
[28] See Parker, S. C., *op. cit.* [25], pp. 274 ff.

[29] *Ibid.*, pp. 428 ff.

With incredible patience, and in appalling detail, these ideas were presented in Pestalozzi's *ABC of Perception* (1803).[30] Its fundamental idea was that of proceeding, step by step, from very elementary to more complex forms, through the three stages of (1) accurate perception, (2) careful oral description, and (3) a painstaking development of clear concepts. Here we encounter Pestalozzi's most serious psychological blunder, that of dissecting a subject into its "simplest elements" and then "combining" these gradually into a connected and symmetrical whole.

In geometry this error proved to be a fatal one, for Pestalozzi selected as his point of departure a minute study of the *straight line* and the *square*.

A sample of Pestalozzi's method. A single example based on Pestalozzi's *ABC of Perception* may serve to give a conception of the wearisome formalism to which he was led by his faulty psychological theories.



Referring to a figure like the one shown at the right, the teacher was expected to conduct oral exercises such as the following, each statement being repeated by the class in unison, and for *each* of the ten lines:

1. This is the *first* (the second, the third . . .) horizontal line.
2. The *first* horizontal line is shorter than the *second*; the *second* is longer than the *first*, but shorter than the *third*.
3. The *first* line is *not* divided; the *second* is divided by *one* point into *two* equal parts.
4. Each of the *two* (three, four, . . .) equal parts of the *second* (third, fourth, . . .) line is a *half* (third, fourth, . . .) of the whole line.
5. The distance from the beginning of the *fourth* (fifth, sixth, . . .) line to the first point on the line is *one-fourth* (-fifth, -sixth, . . .) of the whole line.
6. The first line is one-half the second; the second is twice as long as the third part of the third.

These same statements, 376 in all, are then to be applied to *vertical* lines, "until the children can give every relation with complete ease." A similar discussion of these topics follows: (1) horizontal parallel lines; (2) vertical parallel lines; (3) a single right

[30] A summary of this classic document is given in Treutlein, P., *op. cit.* [23], pp. 15 ff., with an appreciative discussion of its great influence; see also Timerding, H. E., *op. cit.* [23], pp. 10 ff.

angle: (4) two right angles; (5) four right angles; (6) a square (with twenty-seven special exercise[illegible]); (7) a rectangle placed vertically; (8) a rectangle placed horizontally. A consideration of subdivided squares and rectangles closes the preliminary course. All this oral work was to [illegible]ccompanied by related drawing exercises.

Gradual correction of Pestalozzi's mistakes. In reality, *Pestalozzi's aim was to establish the rectangle and the square as the dominant figures of intuitive geometry.* That was a sound idea. However, his monotonous, pedantic procedure ruined everything and caused a great revulsion of feeling on the part of the public. Pestalozzi knew too little about geometry to apply his own maxims profitably in this field, and although some of his immediate followers [illegible] added other "elements," such as triangles and circles, they retained the same barren sequence of points, lines, and combinations of lines, and the same mechanistic formulation.

A great change for the better came about when a friend of Pestalozzi, Karl von Raumer, noted as a historian of education and as a geologist, published his *ABC of Crystallography* (1820),[illegible] through which he opened a new field for geometry as a school subject. He suggested the use of *solids* from the very beginning, recommending for the purpose the most common crystals.

It was Dr. W. Harnisch who carried out this idea in his book, *Die Raumlehre oder die Messkunst, gewöhnlich Geometrie genannt* (Breslau, 1821).[illegible] The solids on which his course was based were the *cube*, the prism, the cylinder, the pyramid, the cone, the regular solids, and the sphere. From these solids Harnisch gradually derived the idea of dimensions, the various angles, and the important plane figures. His course included a study of the fundamental constructions, of mensuration, and of similarity. Above all, Harnisch began his work with oral exercises pertaining to *actual objects* in the schoolroom. He made constant use of models, and created a more natural geometric sequence. His main objective was *practical utility*. In general, Harnisch foreshadowed the substance of what is now taught of geometry in the German elementary schools. The

Many attempts were made to improve Pestalozzi's *ABC of Perception*. For numerous illustration[illegible] and references, see Treutlein, *op. cit.* [illegible], pp. 20-35.

See Treutlein, P., *op. cit.* [illegible], p. 35, also Timerding, H. E., *op. cit.* [illegible], p. [illegible]

Treutlein, P., *op. cit.* [illegible], pp. [illegible], Timerding, H. E., *op. cit.* [illegible], p. [illegible]

obvious defects of that system, which have remained in existence to this day, will be considered on a later page (see page 133).

Harnisch was followed by a number of great methodologists, such as Diesterweg,[54] who wrote several influential treatises on the teaching of geometry, and Kehr,[55] whose *Practical Geometry* is still regarded as a model by present-day writers.

Herbart and the Herbartians.[56] While Pestalozzi's work was, after all, crude and empiric, Herbart (1776-1841) became the logician and philosopher of the new movement. He tried to create a really scientific basis for all education. He attached great importance to *pure mathematics*. Having visited Pestalozzi in 1799, he tried to make the new ideas known by writing two interesting monographs. One of these set forth Herbart's ideas on a first course in geometry.[57] He affirmed that "training in intuition (*Anschauung*) is the peculiar function of mathematics." In place of Pestalozzi's squares and rectangles, he insisted on the *triangle* as the fundamental figure of geometry. The work which he outlined in great detail was in reality a preparatory course in *numerical trigonometry*, intended for children nine or ten years of age. Naturally, these speculative suggestions were so impractical that they were never taken seriously.

On the other hand, Herbart's ideas on general *method* and on the *objectives of education* were destined to exert a vast influence on elementary and secondary teaching in both European and American schools. The later Herbartians (especially Ziller and Rein) developed such doctrines as the correlation theory, the culture-epochs or recapitulation theory, and above all the famous "five formal steps" of teaching. "Genetic" methods began to replace dogmatic, imitative procedures, a fact which has permanently affected the teaching of elementary geometry. In spite of the cogent criticisms directed

[54] Diesterweg, F. A. W., *Leitfaden für den ersten Unterricht in der Formen-, Grössen-, und räumlichen Verbindungslehre*, Elberfeld, 1822.

[55] Kehr, C., *Die Praxis der Volksschule*, pp. 250-261, Gotha, 1877; see also Kehr, C., *Praktische Geometrie für Volks- und Fortbildungsschulen*, fifth edition, Gotha, 1896; tenth edition by Saro, Gotha, 1910. This text furnished the basis for Paul H. Hanus' *Geometry in the Grammar School* [111].

[56] See Parker, S. C., *op. cit.* [251], Chap. XVII; also Monroe, P., *op. cit.* [271], pp. 622-630.

[57] See *Herbart's ABC of Sense Perception*, translated by Eckoff, W. J., D. Appleton and Company, New York, 1896. The German original appeared in 1802. An appreciative discussion and a summary of this interesting book are given by Treutlein, P., *op. cit.* [331], pp. 10-20.

against the Herbartians by modern educational thinkers, especially by John Dewey,[38] the Herbartian movement appears to have left a lasting impression on the teaching of elementary mathematics.

Contributions of the Herbartians. Lack of space prevents a detailed description of the numerous excellent contributions made by the Herbartians in the field of intuitive geometry. The most noted of these are the works of Pickel, Wilk, Martin and Schmidt, and Zeissig, a group of masterful teachers. Being a firm believer in the culture-epochs theory, Dr. E. Wilk wrote a brilliant foundational study,[39] developing this theory in the field of elementary geometry. He traces the genesis of geometric ideas in primitive societies through the instrumentality of the manual arts, sketches the evolution of geometric knowledge through the ages, derives a body of significant objectives from this background, and finally offers a penetrating critique of current tendencies. Wilk's work antedates that of Branford and is perhaps the most thought-provoking single monograph on the significance of intuitive geometry that has ever been written.

E. Zeissig was the most prolific writer of this group.[40] In one of his publications he refers to no less than sixty-seven essays or studies on intuitive geometry as coming from his own pen. His detailed pedagogic analysis of practically every phase of the subject, and his minute lesson plans represent a gold mine of valuable suggestions.

From an American standpoint, however, the most original contribution is that of Martin and Schmidt.[41] They build their entire course around geometric centers of interest, which really represent *form projects*. The first project is the *home*. It furnishes the data for a preliminary study of the rectangle, of parallel and perpendicular lines, of areas and volumes, of isosceles and equilateral triangles, and of the fundamental constructions. The other geometric projects center around the *church* (columns, arches, circles), the *farm* (men-

[38] See the *Second Yearbook of the National Herbart Society*, 1896, for the most thorough discussion of this subject in English; also Parker, S. C., *op. cit.* [25], pp. 422 ff.

[39] Wilk, E., *Der gegenwärtige Stand der Geometrie-Methodik*, Dresden, 1901.

[40] See Zeissig, E., *Zur Reform des Geometrieunterrichts in der Volksschule*, Annaberg, 1894; also *Präparationen für Formenkunde*, in 2 vols., Langensalza, Vol. I, 1897, Vol. II, 1900; and *Formenkunde als Fach*, Dresden, 1899.

[41] Martin, P. and Schmidt, O., *Raumlehre*, in 3 parts, Berlin, 1896-1898; also, *Soll die Raumlehre im Anschluss an einheitliche Sachgebiete behandelt werden?* Dessau, 1899. The latter is a discussion of underlying theories.

suration), the *shop, transportation*, and the *earth* (mathematical geography). Needless to say, this type of organization found few imitators.

It is characteristic of these Herbartians that they derive all geometric information from the child's environment, or from forms suggested by nature and art, that the universality of space relationships is stressed continuously, and that at all times there is an appeal to the learner's interest and self-activity. That is, artificial models are rejected in favor of either *natural objects* or other *real objects* of everyday interest (buildings, rooms, bricks, towers, monuments, clocks, coins, pictures). Models are used only as substitutes when necessary. Much attention is given to geometric ornament and to drawing. Thus, Fresenius [12] characterized geometry as the "grammar of nature."

Altogether, the Herbartian influence made geometry an independent school subject in the elementary curriculum, determined its objectives and its content, and created a very rich literature dealing with its methodology.

The movement led by Fröbel.[13] Whereas the Herbartians had perfected the technique of instruction, thus glorifying the function of the *teacher*, the Fröbelian movement exalted the importance of the *child*. It emphasized the spirit, purpose, *atmosphere*, and *morale* of the schoolroom. Henceforth the school was to give constant attention to "the interests, experiences, and *activities* of the *child*, as the starting point and means of instruction."

Fröbel had spent two years at Yverdon (1808-1810), studying Pestalozzi's methods under the daily guidance of the master himself. His own program, as incorporated in *The Education of Man* (1826) [14] and in *Education by Development*,[15] for many years became identified exclusively with education in the kindergarten. Increasingly however, Fröbel's ideas are seen to be fundamental to *all* stages of education, as he himself had claimed. In fact, the most recent psychological and educational theories of our times are in

[12] Fresenius, K., *Die Raumlehre, eine Grammatik der Natur*, Frankfurt, 1853; second edition, 1875.

[13] See Parker, S. C., *op. cit.* [25], Chap. XVIII; also Monroe, P., *op. cit.* [27], pp. 630-667.

[14] Fröbel, F., *The Education of Man*, translated by Hailmann, William N., D. Appleton and Company, 1894.

[15] Fröbel, F., *Education by Development*, American edition, translated by Jarvis, Josephine, D. Appleton and Company, New York, 1902.

complete accord with Fröbel's major thesis. Fröbel regarded the child as a *creative* rather than a receptive being. *Self-activity* of the mind became the first law of instruction.

As interpreted and transformed by Colonel Parker, William James, John Dewey, William H. Kilpatrick, and others, the essential Fröbelian ideas have greatly influenced the educational philosophy of our American schools.[16]

In the education of little children Fröbel depended on a peculiar symbolism of form, involving the sphere, the cube, and the square. His fifth "gift" involved the construction of geometric forms and "forms of beauty" with square blocks and splints. These ideas were popularized in America by Superintendent W. N. Hailmann.[17]

As to the teaching of intuitive geometry, the influence of the Fröbelian "activity" idea has been apparent on all sides, especially in Germany. *Arbeitsunterricht* (teaching through active participation) is the current slogan in the German schools, as it is with us. It is no longer a question of merely providing *models* or *objects* for exercises in *perception* (the original meaning of *Anschauung*). A *laboratory technique* is made to prevail in the classroom. The pupil himself initiates, observes, creates, and formulates principles through the medium of properly guided and systematized classroom activities and discussions. All concepts are derived as naturally and spontaneously as possible, life relationships and applications are stressed at every point, interest and effort are unified, constructive expression is encouraged, and a gradual growth in mathematical power is effected through the development of genuine insight.

B. THE SECONDARY SCHOOLS[18]

The early period. The first geometry text written in German dates from the end of the fourteenth century and is known as *Geometria culmensis*.[19] It was merely a brief account of rules and

[16] See Monroe, P., *op. cit.* [27], pp. 667-673, for a summary of the educational effects of the psychological movement sponsored by Pestalozzi, Herbart, and Fröbel.

[17] See Parker, S. C., *op. cit.* [43], pp. 438 ff.; also Hailmann, William N., *Constructive Form Work*, C. C. Birchard, Boston, [illegible].

[18] See Stamper, Alva W., *op. cit.* [56], pp. 54-73; Cajori, Florian, in *op. cit.* [18], pp. 13-19; Klein, F. and Schimmack, R., *Vorträge über den mathematischen Unterricht an den höheren Schulen*, pp. 67-69, Leipzig, 1907; Treutlein, P., *op. cit.* [64], pp. 27 ff.; Lietzmann, W., *op. cit.* [34], Vol. I, pp. 211-231.

[19] See Tropfke, J., *Geschichte der Elementar-Mathematik*, Vol. IV, p. 12, second edition, Berlin, [illegible].

practices employed in connection with *surveying*. The purely practical character of this treatise was symbolic of a trend that had persisted through the centuries. That is, the *geometry of everyday life*, as developed by the Egyptians and the Romans, had continued to exist independently of the "scientific" geometry of Euclid. This fact was destined to affect the mathematical work of the secondary schools. It created a dualism of objectives and methods which has been at the bottom of many conflicting tendencies to our day. It is significant that the early *printed* European textbooks in geometry were concerned with the *practical* aspects of the subject. Such books were quite common in the sixteenth and seventeenth centuries. Of particular interest, in this connection, is a geometry written by the great artist Albrecht Dürer, with applications in architecture and in perspective.[30]

The long struggle between the practical and the Euclidean types of geometry in the secondary schools of Europe has been sketched so thoroughly by others that we shall limit ourselves to a few helpful references to the literature on the subject.[31]

It is sufficient to point out that as early as the seventeenth century, geometry was taught in a considerable number of German secondary schools. At first, the work was of a distinctly practical nature. The transition to a more scientific type of geometry was due to the influence of men like Leibnitz (1646-1716), Christian Wolff (1679-1754), and Kästner (1719-1800). They prepared the way for a formal, "disciplinary" organization of the mathematical curriculum, which prevailed during most of the nineteenth century.

Recent developments. Around 1800 the doctrines of Pestalozzi and Herbart began to be felt in the secondary schools. Genetic (heuristic) methods began to replace the traditional Euclidean technique. At about the same time, the demand arose for a "propaedeutic" (preparatory) course in geometry. For decades this question was agitated. There were violent partisans in favor of and opposed to this innovation.[32] At last a complete victory re-

[30] Dürer, Albrecht, *Underweysung der Messung mit dem Zirckel und Richtscheyt in Linien, Ebenen und gantzen Corporen*, Nuremberg, 1525, with Latin editions, Paris, 1532 and 1535, and Arnheim, 1605.

[31] In addition to the sources F... previously, see, for example, Sanford, Vera, *op. cit.*, 1927, Chap VI, also Kokomoor, F. W., "The Teaching of Elementary Geometry in the Seventeenth Century," and "The Distinctive Features of Seventeenth Century Geometry," *Isis*, Vols X and XI.

[32] For many interesting details, see Schotten, H., *Inhalt und Methode des planimetrischen Unterrichts*, Leipzig, 1890.

sulted in favor of systematic preparatory work in geometry. The culmination of this movement was reached in the famous *Meran Report* of 1905, prepared by a committee of twelve distinguished scientists and mathematicians.[] Its principal demand was that of continuous "training in space intuition and in functional thinking" throughout the secondary period.

This report, as amended and revised in later editions, is now being followed by a majority of the current textbooks and courses of study. The essential features of the newer curricula will be considered in a later section.

III. Intuitive Geometry in England, France, and Italy

A. England

The Perry Movement. The great agitation of 1901, known as the "Perry Movement," referred to above, became of epoch-making importance not only in England but also in America.[] Among the suggestions, old and new, that were stressed by Perry and his followers, the following are of significance in this connection:

1. Experimental geometry and practical mensuration should precede demonstrative geometry.
2. Intuition, measurement, motion, visual devices, graphic methods, and squared paper should be used throughout.
3. Practical applications should be stressed.
4. Work with solids should be introduced very much earlier.
5. Some deductive reasoning should accompany experimental geometry.
6. Laboratory techniques should be encouraged.

The extreme demands of Perry were soon rejected. A middle ground was gradually evolved which may be inferred from such texts as those of Barnard and Childs[] and of Godfrey and Siddons.[] A

Detailed summaries or discussions of this important document are given by Treutlein, P., *op. cit.* [23]; Klein, F. and Schimmack, R., *op. cit.* [48], pp. 208-220; Höfler, Alois, *op. cit.* [23]; Lietzmann, W., *op. cit.* [23], Vol. I, pp. 226 ff.

Young, J. W., *The Teaching of Mathematics*, Chap. VI, Longmans, Green and Co., New York, 1924; also, *Final Report of the National Committee of Fifteen on Geometry Syllabus*, pp. 27 ff., *op. cit.* [1].

Barnard, S. and Childs, J. M., *A New Geometry for Schools*, The Macmillan Company, New York, 1903.

Godfrey, C. and Siddons, A. W., *Elementary Geometry*, Cambridge University Press, Cambridge, 1903.

pioneer book of this period was a text prepared by Hamilton and Kettle.[57] A more conventional treatment is that found in an introductory course prepared by Hall and Stevens.

Of great significance was the work of Benchara Branford, who outlined a historical program[58] similar to that of the Ziller school in Germany. His own "first lessons" (pages 13-22) would hardly be imitated by anyone to-day. This is also true of his lack of system. But the multitude of valuable suggestions distributed throughout the book make it a source of inspiration even for our more systematic courses. Influenced by the reading of Branford's manuscript, David Mair issued his *School Course of Mathematics*,[59] equally unsystematic, but full of excellent ideas. It is based on progressively arranged problems and demands much independent thinking on the part of the learner.

A most interesting and original contribution of the same period is a delightful booklet by Mrs. Edith L. Somervell, entitled *A Rhythmic Approach to Mathematics*.[60] It should become more widely known. It shows how children of the preschool age can be introduced to geometric designs by a series of manual activities of amazing power and charm. In that connection, attention should be called to T. Sundara Row's *Geometric Exercises in Paper Folding*.[61]

In 1905 there appeared the *First Book of Geometry* by Dr. Grace Chisolm Young and Dr. W. H. Young.[62] It was intended for children as young as seven years of age, working with or without a teacher. Based almost entirely on the method of *paper folding*, it has become an educational classic of its kind. It covers a very large territory, including all the important plane and solid figures. Among its obvious defects is its abstract character and the absence of practical problems.

[57] Hamilton, J. G. and Kettle, F., *A First Geometry Book*, seventh edition, London, 1909.

[58] Branford, Benchara, *A Study of Mathematical Education*, pp. 13-22, Oxford, 1908; new edition, 1924. An excellent German translation appeared in 1913.

[59] Mair, David, *School Course of Mathematics*, Oxford, Clarendon Press, 1907.

[60] Somervell, Edith L., *A Rhythmic Approach to Mathematics*, London, 1906.

[61] Row, T. Sundara, *Geometric Exercises in Paper Folding*, American edition, prepared by Beman, W. W. and Smith, D. E., The Open Court Publishing Company, Chicago, 1901.

[62] Young, Grace Chisolm and Young, W. H., *First Book of Geometry*, London, 1905. A very satisfactory German translation appeared in 1908 under the title *Der kleine Geometer*.

Later Developments. In August 1912 the International Congress of Mathematicians met in Cambridge, England. One of its sessions was devoted to a discussion of "Intuition and Experiment in Mathematical Teaching in the Secondary Schools." Professor David Eugene Smith presented a report on the subject, which was followed by a discussion participated in by representatives of Germany, Austria, France, and Great Britain. The printed summary published in *l'Enseignement mathématique* (November, 1912) contains numerous valuable suggestions that might well be restated even at this time.

A critical discussion of the rôle of intuition in geometry was offered in 1913 by G. St. L. Carson.[63] The distinction between "deduction" and "proof," which he insists on, is of importance because "pupils can appreciate and obtain proofs long before they can understand the value of deductions," and "because the functions of proof and deduction are entirely different." Of interest, also, is Carson's critique of exercises in drawing and measurement. A liberal number of assumptions is advocated. He says that "in the earlier stages every acceptable statement or intuition should be taken as an assumption; the analysis of these, to show on how small an amount of assumption the science can be based, being deferred."[64]

The recent period. In England appeared the well-known report on *The Teaching of Geometry in Schools*,[65] prepared for the Mathematical Association (1923). It is comparable in importance only to the *Meran Report* of Germany (1905) and the report of the National Committee on Mathematical Requirements in the United States (1923). It states correctly that "a great deal of school geometry is a compromise." The most outstanding feature of the report is Part II, which deals with the "division of the geometry course into stages." No less than *five* stages are enumerated and described. Stage A is designated as the "experimental stage," which "should end at the age of about 12½ years." Stage B is the "deductive stage." At this stage the deductive method should be "prominent, but not to the exclusion of intuition and induction." "In Stage A the deductive method is weakly developed, but in

[63] Carson, G. St. L., *Essays on Mathematical Education*. Ginn and Company, London and Boston, 1913.

[64] *Ibid.*, p. 30.

[65] *The Teaching of Geometry in Schools*, a report prepared for the Mathematical Association. London, first edition, 1923, second edition, 1925.

Stage C it is supreme." The nature and content of Stage A will be outlined in a later section of this monograph (see pages 126 ff.).

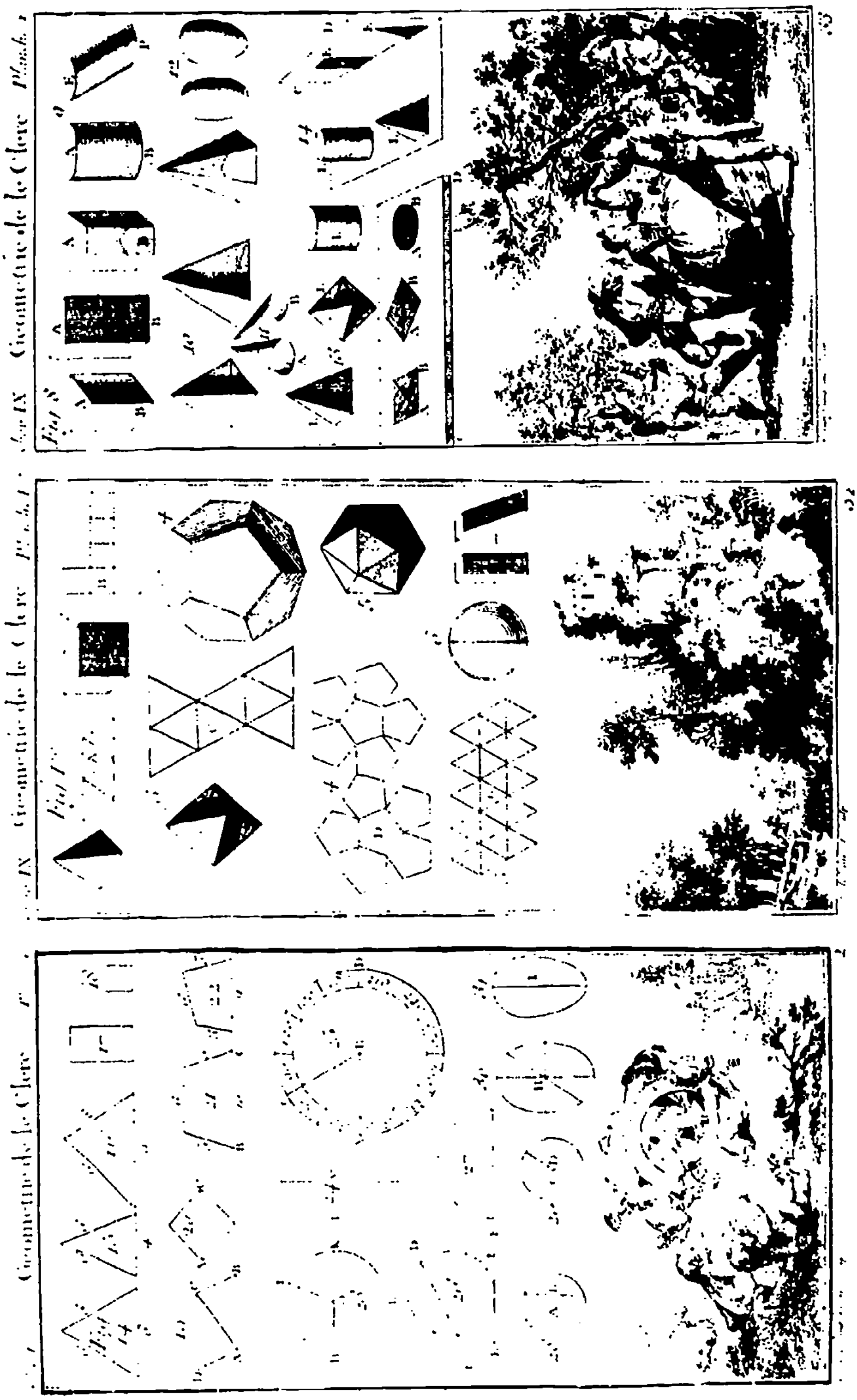
B. FRANCE

The early period. A critical attitude toward Euclid was shown by French writers from the very beginning. Practical geometry was given as much emphasis as in Germany. The first geometry printed in French, that of Bouelle,[64] contained two chapters on the relation of geometry to *symmetry* as seen in the animate and inanimate world. The book was not rigorous, some theorems being mere statements of facts.

Allain Manesson Mallet. *la Géométrie pratique.* T.II. 1702.

Petrus Ramus (1580) treated geometry as the art of accurate measurement. Le Clerc's fanciful geometry (1730) extended the idea of Comenius' *Orbis Pictus* by supplying numerous pictures which were supposed to make the propositions real. No proofs were given.

[64] Interesting data concerning the early French texts may be found in Stamper, Alva W., *op. cit.* [22], pp. 73-87; Cajori, Florian, in *op. cit.* [18], pp. 6-13; Kokomoor, F. W., *op. cit.* [51]; the standard histories of mathematics.



SEBASTIEN LE CLERC. *Traité de géométrie.* 1739.

Geometry began to be studied in the French secondary schools during the first half of the eighteenth century. The greatest influence on the teaching of geometry was exerted by Legendre's *Éléments de géométrie*, which appeared in 1794. It was widely imitated, not only in Europe, but also in the United States.[67] It abandoned the sequence of Euclid, made greater use of intuition, emphasized solid geometry, and admitted the use of algebraic methods. It was "the first logical departure from Euclid that the world recognized."[68] The ideas of Legendre have survived in French and American texts to this day.

Recent developments. For many years France has had the good fortune of seeing some of her most eminent mathematicians interested in the preparation of texts for her secondary schools. To prove this, it is only necessary to point out authors like Hadamard, Rouché and Comberousse, Borel, Bourlet, Méray, Niewenglowski and Gérard, and Tannery. This fact has exerted a profound influence on all mathematical instruction in France. It is characteristic of the French secondary schools that instruction is given in three successive stages or "cycles," comprising, respectively, four years, two years, and one year.[69] Each cycle begins by reviewing all the mathematics of the preceding cycle, but on a higher level. In the first cycle, *intuition* predominates, and only gradually is the *deductive* method allowed to rule.

In geometry, the most noteworthy recent tendencies in France have centered around the use of "motion," the fusion of plane and solid geometry, and extensive development of geometric drawing.

In the elementary field, the available literature is not remotely comparable with that of Germany. Attention should, however, be directed to a fascinating little book by Professor C. A. Laisant of the École Polytechnique in Paris, entitled *Initiation mathématique*.[70]

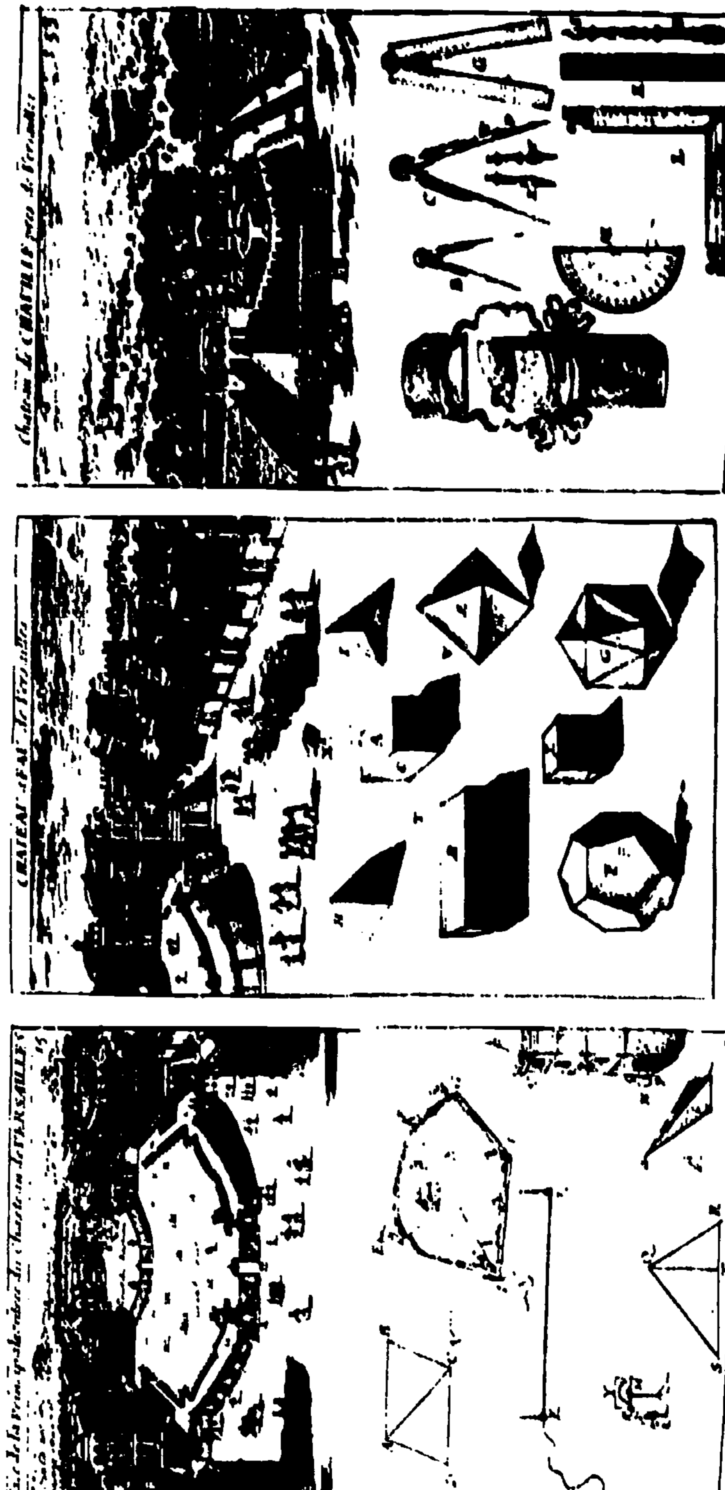
[67] See, for example, Davies, Charles, *Elements of Geometry and Trigonometry*, from the works of A. M. Legendre, A. S. Barnes & Company, New York, 1862. The first paragraph of the preface reads as follows:

"Of the various Treatises on Elementary Geometry which have appeared during the present century, that of M. Legendre stands preëminent. Its peculiar merits not only have won for it a European reputation, but have also caused it to be selected as the basis of many of the best works on the subject that have been published in this country."

[68] Stamper, Alva W., *op. cit.* [20], p. 82.

[69] Klein, F., *op. cit.* [21], p. 283.

[70] Laisant, C. A., *Initiation mathématique*, Genève-Paris, 1906; German translation, *Einführung in die Mathematik*, Leipzig, 1908; American edition, *Mathematics*, Doubleday, Page and Co., New York, 1914



ALLAIN MANESSON MALLET. *la Géométrie pratique.* T.I. 1702.

It is described by the author "as a work without any program, dedicated to the friends of children." Intended for children "between the ages of 4–11," it offers a series of informal mathematical games, discussions, and problems covering a very wide range. Many of its ideas are of permanent value.

An interesting *Géométrie, cours élémentaire*, published in Tours and Paris (undated) by a committee of teachers, shows a wealth of illustrations, pictures, designs, and concrete applications that make some of its pages most attractive. In very condensed form it presents the fundamental concepts, as well as the typical facts relating to plane figures and solids. In conclusion, it offers some fine work in outdoor measurement. A second course, by the same authors, presents the main ideas and applications of plane and solid geometry in equally simple form.

Another brief but valuable booklet was prepared by Professor Paul Bert, as early as 1886.[71] The preface states that this little book is "both a preparation for more regular and more advanced study, and at the same time it is a definite and complete work for the great bulk of our pupils in our elementary schools; a work, that is, which is self-sufficing, complete in itself." The entire course is built up on direct and indirect measurement, much attention being given to *outdoor exercises in surveying*.

In the elementary schools of France, the study of geometry parallels that of drawing, supplemented by the most practical rules of mensuration. The revision of the curriculum made in 1923 produced no marked changes in the geometric work.

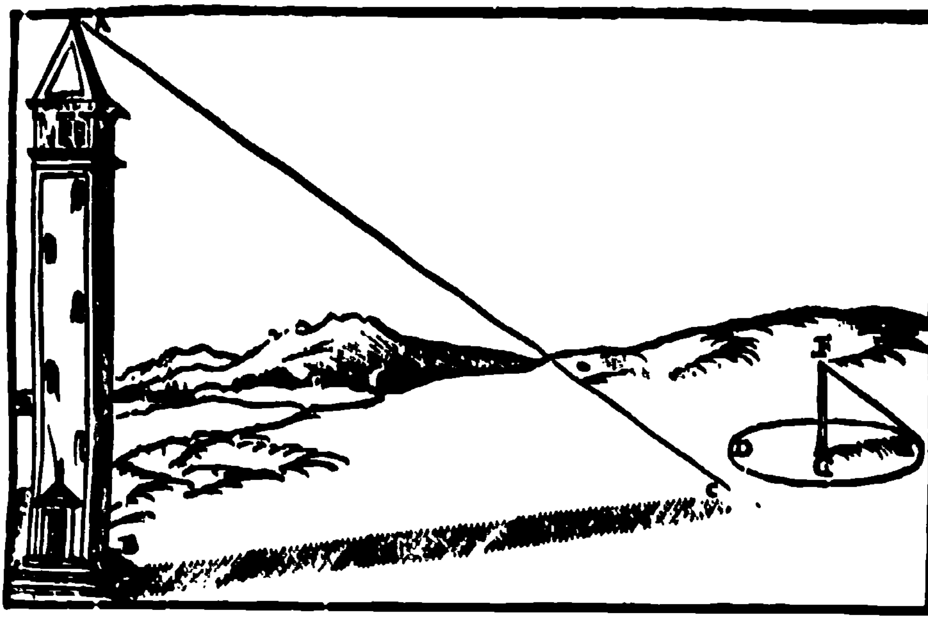
C. ITALY

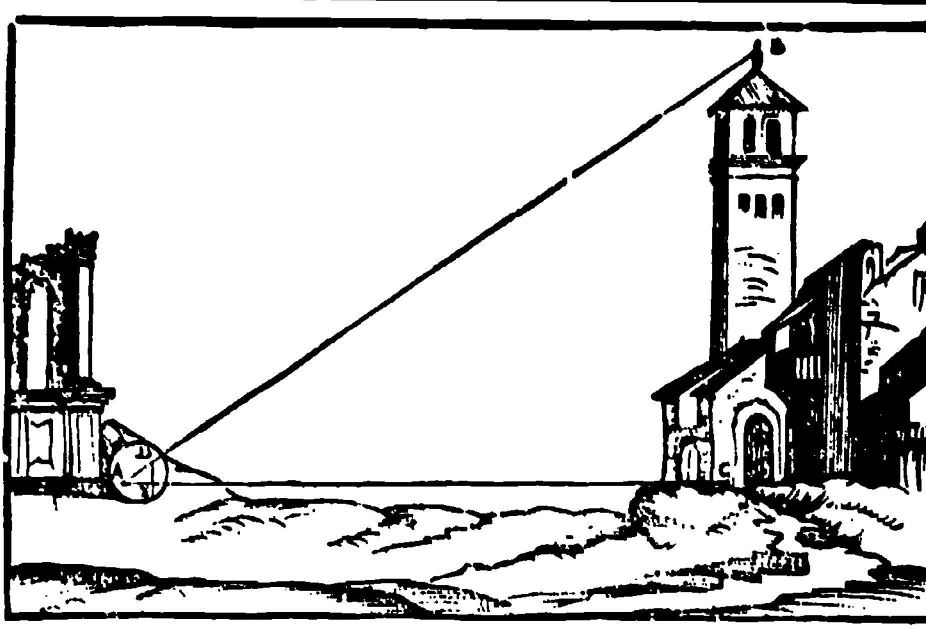
The early period. The first printed edition of Euclid, the Campanus translation, was issued in Venice, in 1482. A few years later (1494), Luca Pacioli's *Summa* appeared, a general compilation of the mathematical knowledge of the time, including an unsatisfactory summary of Euclid. As in the case of Germany and France, practical geometries were common in Italy in the sixteenth century. Cosimo Bartoli (1503-1572) wrote a popular book on mensuration.[72] Silvio Belli's practical geometry (1505)[73] deals primarily with sur-

[71] Bert, Paul, *First Elements of Experimental Geometry*, English translation, London, 1886.

[72] Stamper, Alva W., *op. cit.* [illegible], p. 57; also Smith, D. E., *History of Mathematics*, Vol. I, p. 308, Ginn and Company, Boston, 1923.

[73] Smith, D. E., *op. cit.* [illegible], p. 304; Stamper, Alva W., *op. cit.* [illegible], p. 57.





SILVIO BELLI. *Quattro libri geometrici.* 1595.

veying problems. On the whole, however, the geometric traditions in Italy were Euclidean in character.

Revival of Euclidean rigor. On the recommendation of Cremona, in 1867, the Italian Government caused a complete return to Euclid. The edition of the *Elements* prepared by Brioschi and Betti (1867) helped to make this restoration effective.[54] Since then, a series of ultra-rigorous textbooks have appeared in Italy, which aim to incorporate in the geometry curriculum of the secondary school some of the latest refinements resulting from modern research on the foundations.[55] According to Klein, the excessively scientific atmosphere of these books is undoubtedly too rarefied for all but a few specially gifted minds. It is also noteworthy that Italy, for a time, gave considerable support to the plan of fusing plane and solid geometry.

The effect of the Gentile Reform of 1923 is described by Professor F. Enriques in the *Fourth Yearbook* of the National Council.[56]

Intuitive geometry in Italy. Several types of elementary texts have appeared in Italy which represent widely varying interpretations of the objectives of a first course in geometry. In 1901 Professor P. Pasquali issued his *Geometria intuitiva.*[57] It is "addressed to the eye and the hand." *No geometric instruments are used. Paper folding* is the chief tool for introducing and constructing the principal plane figures and dealing with their properties. One section is definitely devoted to an analysis of "Fröbelian foldings." It is highly instructive to see how the Pythagorean theorem is presented, how algebraic relations are pictured, and how regular polygons and complicated relationships are handled by this simple manual technique.

On the other hand, in Professor Veronese's *Nozioni elementari di geometria intuitiva,*[58] we find a rather rigorous outline of the fundamentals. The greatest care is taken in formulating accurate

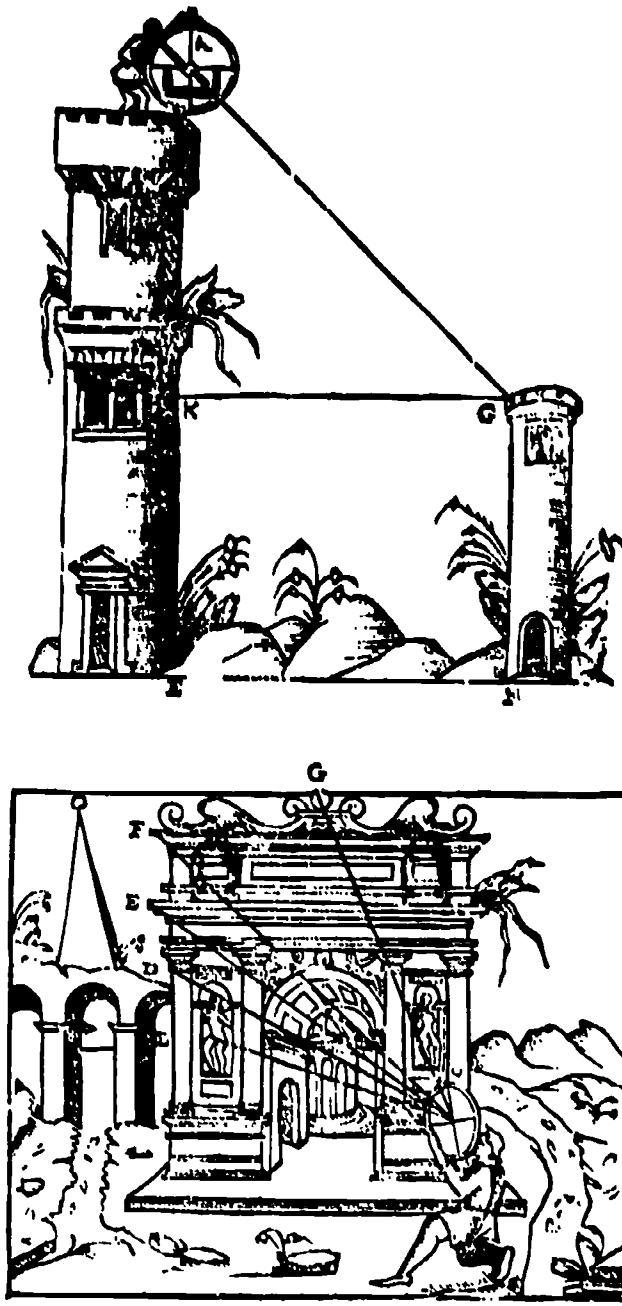
[54] *The Fourth Yearbook, The National Council of Teachers of Mathematics*, 1929, pp. 71-76.

[55] See, for example, the texts of Sannia and d'Ovidio, De Paoli, Faifofer, Veronese, Ingrami, and Enriques and Amaldi. Some of these treatises (Veronese, Ingrami) follow a fusion plan. The fusion movement encountered difficulty and appears to have been abandoned in recent years.

[56] Enriques, F., article in *The Fourth Yearbook, The National Council of Teachers of Mathematics*, 1929, pp. 71-75.

[57] Pasquali, P., *Geometria intuitiva*, in 3 parts, Parma, 1901.

[58] Veronese, G., *Nozioni elementari di geometria intuitiva*, third edition, Verona, 1906.



Cosimo Bartoli. *Del modo di misurare.* 1589.

concepts and definitions. Rules of mensuration, as well as the fundamental constructions, are given without proof. There are very few applied problems, and no references to life situations. Only eight pages are devoted to solids.

Whereas the booklets mentioned above are virtually limited to plane figures, the *Geometria intuitiva* of Dr. G. Costanzo and C. Negro,[77] intended for the first three classes of the gymnasium, has as its outstanding feature a *fusion of plane and solid geometry*. Thus, a discussion of parallel lines is at once followed by a consideration of parallel planes. There are no formal proofs, either of propositions or of constructions. The course closes with "developments" of solids, including the five regular solids.

IV. Intuitive Geometry in the United States

The early period. Reference has already been made to the pioneer efforts of Thomas Hill, Bernhard Marks, and G. A. Hill. The *First Lessons* of Thomas Hill really represent a series of very elementary conversations about the purpose of geometry and about the principal plane figures and their properties, followed by a brief discussion of the cube, the cone, and the sphere. Its most ingenious feature is an informal treatment of certain higher plane curves, such as the conic sections, the cycloid, and the catenary.

Marks's book, on the other hand, is much more conventional. Part I consists of twenty-seven "lessons" on the *concepts* relating to lines, angles, triangles, and other plane figures. Much of the material is given in question form and is very tiring to a modern reader. There are practically no applied problems and no realistic illustrations. Part II contains twenty-five propositions which are demonstrated in considerable detail, many of the proofs being preceded by a "development lesson." An interesting feature of the book is the use of color in some of the diagrams.

G. A. Hill's *Geometry for Beginners*, a substantial volume of 314 pages, is one of the most ambitious texts of its kind ever prepared. Its thirteen chapters cover practically the whole field of everyday geometry, including a whole chapter on the ellipse. The treatment is very systematic. The sequence is as follows: introduction, straight lines, angles, triangles (including symmetrical figures), areas (Pythagorean theorem, transformation and "partition" of

[77] Costanzo, G. and Negro, C., *Geometria intuitiva e rudimenti di disegno geometrico*, second edition, Bologna, 1907.

figures included) similar figures, the circle, the ellipse, planes, "geometrical bodies" (prism, cylinder, pyramid, cone, sphere), surfaces, and volumes. Numerous exercises are given, and the practical uses of geometry are stressed. Much of this volume would now be omitted in a first course, but its general point of view was far ahead of its time. The following sentence from the preface is of interest:

> The author, while residing in Germany in 1877 and 1878, made himself familiar with their methods and textbooks, and he has freely used the knowledge thus acquired in writing the present work.

A few years later, the same author issued his *Lessons in Geometry for the Use of Beginners.*[50] It was "prepared to meet the wishes of those who prefer a shorter and easier introductory course in Geometry than that given in the *Geometry for Beginners.*" The author says that he had in mind "pupils between the ages of twelve to sixteen." There are ninety-six "lessons" which cover much the same territory as the earlier book, but in rather formal fashion. There are many exercises and problems. The *cube* is the first figure presented, as in the earlier book. A noteworthy feature is the (optional) use of *metric* units and the attention given to geometric designs. The author says that the lessons are intended for "a course of three hours per week for a year, or, what is better for the pupil, a course of one hour per week for a year, and a course of two hours per week for the year following." He continues: "Geometry, as here presented, should be studied before algebra."

In 1888, W. W. Speer (Cook County Normal School) wrote his *Lessons in Form*, advocating that "form lessons" should precede the direct study of number (page 69). The influence of the German movement is plainly reflected in this book.

Professor Paul Hanus, of Harvard University, in the course of lectures referred to on pages 64 and 65, given in 1893, made known the detailed plans found in Kehr's *Praktische Geometrie*, "for the last three years of the grammar school," together with a brief summary of Kehr's principles of method.

Report of the Committee of Ten (1894). Because of the historic importance of this report, copies of which are no longer available, the section which deals with "Concrete Geometry" is here reproduced in its entirety. It will be seen that we still have

[50] Hill, G. A., *Lessons in Geometry for the Use of Beginners*, Ginn and Company, Boston, 1887.

very much to learn from this forward-looking document. To quote:

The Conference recommends that *the child's geometrical education should begin as early as possible; in the kindergarten, if he attends a kindergarten, or if not, in the primary school.* He should at first gain familiarity through the senses with simple geometrical figures and forms, plane and solid; should handle, draw, measure, and model them; and should gradually learn some of their simpler properties and relations. It is the opinion of the Conference that in the early years of the primary school this work could be done in connection with the regular courses in drawing and modelling without requiring any important modification of the school curriculum.

At about the age of ten for the average child, systematic instruction in concrete or experimental geometry should begin, and should occupy about one school hour per week for at least three years. During this period the main facts of plane and solid geometry should be taught, not as an exercise in logical deduction and exact demonstration, but in *as concrete and objective a form as possible.* For example, the simple properties of similar plane figures and similar solids should not be proved, but should be illustrated and confirmed by cutting up and re-arranging drawings or models.

This course should include among other things the careful *construction of plane figures,* both by the unaided eye and by the aid of ruler, compasses and protractor; the *indirect measurement of heights and distances* by *the aid of figures carefully drawn to scale;* and *elementary mensuration, plane and solid.*

The child should learn to *estimate* by the eye and to *measure with some degree of accuracy the lengths of lines, the magnitudes of angles, and the areas of simple plane figures; to make accurate plans and maps from his own actual measurements and estimates; and to make models of simple geometrical solids in pasteboard and in clay.*

Of course, while no attempt should be made to build up a complete logical system of geometry, the child should be *thoroughly convinced of the correctness of his constructions and the truth of his propositions by abundant concrete illustrations, and by frequent experimental tests;* and from the beginning of the systematic work *he should be encouraged to draw easy inferences,* and *to follow short chains of reasoning.*

From the outset *the pupil should be required to express himself verbally as well as by drawing and modeling, and the language employed should be, as far as possible, the language of the science,* and *not a temporary phraseology to be unlearned later.*

It is the belief of the Conference that the course here suggested, if skilfully taught, *will not only be of great educational value to all children, but will also be a most desirable preparation for later mathematical work.*

Then, too, while it will on one side supplement and aid the work in arithmetic, it will on the other side fit in with and help the elementary instruction in physics, if such instruction is to be given.[31]

[31] *Report of the Committee of Ten on Secondary School Studies,* pp. 110-111 [112].

Later developments (1895-1923). Adelia R. Hornbrook's *Concrete Geometry* (1895),[52] as the author declared, was "designed especially for use in grammar grades in accordance with the recommendations of the Committee of Ten and with the practice of many of our foremost schools." Besides, the book was recommended "for supplementary work for beginners in demonstrative geometry." Its general method is "that of constructing and inspecting geometric forms and of reporting in the language of mathematics the results of the inspection." The pupil's success is said to "depend very largely upon the teacher." The book is essentially an exercise manual covering the whole range of the conventional plane figures, the cube being the only solid receiving attention. Many of the exercises are excellent and stimulating.

William T. Campbell, of the Boston Latin School, in 1899 brought out his *Observational Geometry*.[53] Part I consists chiefly of laboratory work centering around the making of models (cube, prism, pyramid, cylinder, cone, sphere). The plane figures are derived from the solids. There are many pictures and interesting diagrams. Part II presents a more systematic study of points, lines, angles, polygons, circles, constructions, mensuration, similar figures, and surveying. It is a charming and interesting book, in spite of its obvious defects.

In 1901 there appeared the *First Steps in Geometry* by Wentworth and Hill.[54] It contains the best features of Hill's earlier books, is beautifully illustrated, and suggests valuable laboratory exercises. The preface sets forth the program of the authors in these words:

> This book is intended to be an introduction to elementary geometry. It aims to make clear by illustrations, definitions, and exercises the exact meaning of the straight line, parallel lines, *axial and central symmetry*, loci of points, equal figures, equivalent figures, similar figures, and measurements of lines, surfaces, and solids.
>
> It aims also to make the learner familiar with the most important theorems, and to teach him to draw, with instruments and free-hand accurate figures both plane and solid.

[52] Hornbrook, Adelia R., *Concrete Geometry*, American Book Co., Inc., New York, 1895.

[53] Campbell, William T., *Observational Geometry*, American Book Co., Inc., New York, 1899.

[54] Wentworth, G. A. and Hill, G. A., *First Steps in Geometry*, Ginn and Company, Boston, 1901.

William A. Schoch, of the Crane Manual Training High School in Chicago, in 1904 issued an excellent *Introduction to Geometry.* Its spirit is practical and stimulating throughout. It has a splendid chapter on symmetry and valuable work on *scale drawing*, and gives one of the earliest treatments of *graphic representation.*

The *Final Report of the National Committee of Fifteen on Geometry Syllabus* (1912) devotes an entire section to "preliminary courses for graded schools" (pp. 46-49). It asserts that "it is of the utmost importance that some work in geometry be done in graded schools." Helpful suggestions concerning this work are given. Special stress is laid on drawing to scale, on models and "patterns," and on "the forms of solid geometry."

The *junior high school movement*, which began at about this period (1912-1915), absorbed these suggestions and gradually incorporated them in the textbooks that were prepared for the new schools which soon sprang up all over the country.

The recent period. The report of the National Committee on Mathematical Requirements appeared in 1923. It served as a clearing house and a summary of current tendencies, and definitely placed intuitive geometry into the mathematical curriculum of the junior high schools. Since then many variations of the program outlined in that report have been tried, and the end is not yet. In a later section we shall examine some of the syllabi and resulting textbook changes that have been evolved during the recent past.

PART THREE

THE BASIC CONTENT OF INTUITIVE GEOMETRY

Importance of compelling types of motivation In recent years the curriculum experts have suggested various criteria for the selection of the instructional materials which are to appear in a modern course of study. Scores of treatises have presented an imposing body of guiding principles for the elaboration of desirable objectives. Valuable as some of this literature has been, most of it has been too theoretical or too complex to be of much use to the struggling teachers who must accomplish a definite task in a limited

Schoch, W. A., *Introduction to Geometry*, Allyn and Bacon, Boston, 1904.

Reprinted in *The Mathematics Teacher*, Vol. V, pp. [illegible], December, 1912.

See Betz, W., "The Development of Mathematics in the Junior High School," *The First Yearbook, The National Council of Teachers of Mathematics*, 1926, pp. 141-165.

period of time. What is still lacking is a simple but comprehensive point of view, in the light of which all classroom procedures and lesson plans can be evaluated. In other words, for each piece of school work we must have a convincing, a *compelling* doctrine on its basic importance. Only then shall we get away from trivial, arbitrary details. Instead, it must be our aim to anchor each subject on the permanent bedrock of fundamental human needs, interests, and aspirations. Such a basis should be, and can be, created for the subject of intuitive geometry.

I. THE PERMANENT BACKGROUND OF GEOMETRY[55]

The primary sources of geometric knowledge. It has been pointed out (page 58) that arithmetic and geometry owe their origin to two fundamental processes, namely, *counting* and *measuring*, and that all constructive activities involve considerations of *shape*, *size*, and *position*. Now, what caused people to become acquainted with and to cultivate the great range of geometric concepts, skills, facts, and relationships? We shall try to show that *nature and the practical arts are the primary and permanent sources of geometric learning*. Once this fact is generally recognized, we shall have established geometry as one of the great social institutions that can no more be destroyed or overlooked than the force of gravitation.

A. THE STUDY OF SHAPE

How the knowledge of form began. The earliest knowledge of form undoubtedly was obtained from natural objects. Fruits, shells, trees, animals, flowers, crystals—all furnished many beautiful patterns. A pool of water suggested a flat or plane surface. The

[55] Much of the material presented in the following pages is based on previous publications for which the writer was responsible either entirely or in part. Instead of using specific quotations, he wishes to express his particular indebtedness to the following sources:

Betz, W., *Geometry for Junior High Schools*, published for the Board of Education by Ezloff, J. M., St. Paul Street, Rochester, New York, 1925.

Betz-Miller-Miller, *Workbook in Intuitive Geometry*, published by the Harter Publishing Company, Cleveland, Ohio, 1928.

Tentative Course of Study in Junior High School Mathematics, for the junior high schools of Rochester, unpublished, Board of Education, Rochester, N. Y., 1930.

Pupils of the Lincoln School (Lincoln School of Teachers College, Columbia University) *Illustrated Mathematical Talks*, pp. 23-36. (Out of print.) Bureau of Publications, Teachers College, Columbia University, New York, 1920.

sky looked like a hemisphere. The rainbow was a huge circular arch. The disk of the sun and of the full moon resembled circles. A heap of sand looked like a cone. The path of a falling stone appeared to be a straight line. Raindrops were seen to fall in parallel lines. The vertical position of trees called attention to the "right angle." At night the stars, like points of fire, outlined innumerable figures.

In short, nature is a sort of huge museum containing an endless variety of shapes. Says Emerson:

> The eye is the first circle: the horizon which it forms is the second: and throughout nature this primary picture is repeated without end. It is the highest emblem in the cipher of the world. St. Augustine described the nature of God as a circle whose centre was everywhere and its circumference nowhere. We are all our lifetime reading the copious sense of this first of forms.[59]

But it was one thing to *see* these forms, and quite another thing to *use* and to *name* them. Fortunately, the early people were driven to a knowledge of form by sheer necessity. They all had to have food, clothing, shelter, weapons, tools, and simple household furniture. The food had to be stored, and so they made pottery and baskets. Clothing had to be made either of skins, or of wool, or of plant fibers. Shelters had to be built and equipped. It took thousands of years to perfect these household arts. But all these activities led to an increasing knowledge of shape, size, and position, and so prepared the way for our present science of geometry.

For illustrative purposes, we shall confine ourselves to a brief rehearsal of the manner in which *shelter* activities became a source of geometric information.

Early shelters. In olden times people lived in natural shelters such as caves or trees. When they became hunters or took up cattle raising, they had to wander a good deal. This compelled them to "break camp" very often, and then to put up a new shelter each time.

What kind of shelter did they build? That depended largely on the building material which could be obtained, on the climate, and on other items. The most common of the early shelters will be described briefly in the following paragraphs. Fortunately, the home life of the American Indian, on the great western reservations

[59] See Emerson, Ralph Waldo, "Circles," *Essays*, First Series, p. 227, Dodge Publishing Co., New York, undated.

and in South America, still gives us a very good idea of all the practical arts which must have been carried on by wild tribes in other parts of the world many years ago. For that reason, Indian homes furnish the most convenient and interesting examples of primitive buildings for our purposes. In putting together their simple shelters, even the uncivilized people learned much about geometry.

1. *The tipi.* Every American boy and girl has, no doubt, heard of the Indian *tipi*. It has the form of a *cone*. The figure that had to be outlined on the ground in building such a conical shelter was the *circle*. The framework of a tipi consisted of long, slender poles. It was covered with birch bark, skins, branches, or mats. The Indians often pictured a tipi by means of a *triangle*.

2. *The wigwam.* Another kind of shelter, used very widely, had the form of a dome or a bowl turned over. The Algonquin Indians in the eastern part of the United States called such a dome-shaped hut a "wigwam." A wigwam has the form of a *hemisphere*. Again, the figure that had to be outlined on the ground in building a wigwam was a *circle*. When seen at a distance, a wigwam looked like a *semicircle*.

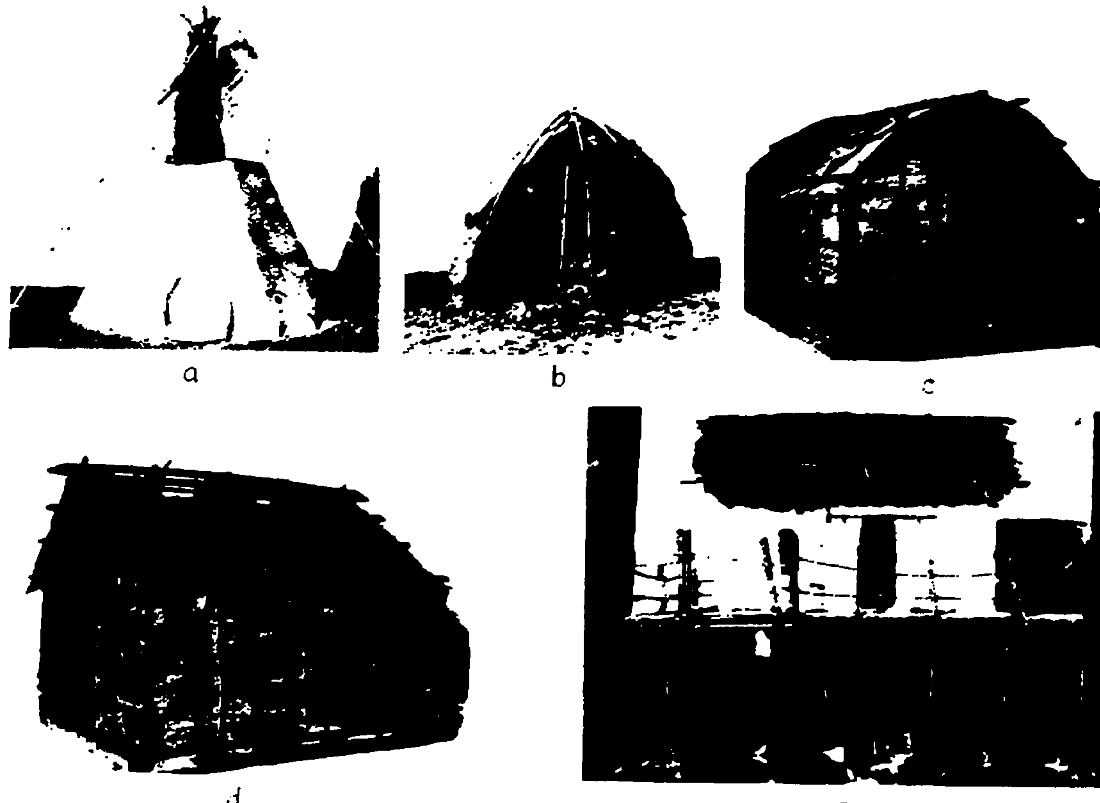
3. *Raised shelters.* It was hard to walk around inside either a tipi or a wigwam. In many parts of the world such huts were therefore raised and put on posts which were set upright in the ground. In some cases a hole was dug under the hut. The space thus added had the form of a *cylinder*. The form that had to be outlined on the ground in making a *cylindrical* hut was a *circle*.

The three shelters named so far are based on the "three round bodies," namely, the *sphere*, the *cylinder*, and the *cone*. They all have a *circle* as their ground plan.

Rectangular shelters. As soon as farming began and people settled down, they put up more permanent shelters or houses which gradually took the forms that are commonly used to-day.

1. *The lean-to.* Hunters and woodsmen often build a "wind-break" having the form of a "lean-to." The ground plan of such a shelter is a *rectangle*.

2. *The gable house.* A lean-to does not afford much protection against the cold or against animals. It was only natural to think of completing it by building a "double lean-to." Such a shelter was used by certain Indian tribes, like the Apache Indians. A shelter of this kind looked like a modern tent, or like a gable



(*a*) Plains Indian tipi. (*b*) Bark sweat lodge, usually a hemisphere. (*c*) Arbor-roofed Iroquois dwelling. (*d*) Angular-roofed Iroquois lodge. (*e*) Swiss lake dwelling.

This illustration is reproduced by the courtesy of Dr. Arthur C. Parker, Director of the Rochester Municipal Museum, who supplied pictures of models made by members of his staff under his supervision.

roof of an ordinary house. The ground plan of a gable house has the form of a *rectangle*. This is also the shape of each of the sloping surfaces or "faces." The front of such a shelter looks like a *triangle*. A form like that of a gable house evidently resembles a *triangular prism*.

3. *The rectangular house*. As in the case of a tipi or a wigwam, it is hard to move about in a tent-shaped house. Gradually people learned how to put the gable house on a framework of poles or to dig a "cellar" under it. The space thus added had the form of a rectangular box or a *rectangular solid*. The rectangular hut came into use in many parts of the world. Obviously, this kind of shelter is the forerunner of the ordinary modern home.

4. *The flat-roofed house*. In regions where there was little or no rain, the gable roof was often omitted, thus leading to a flat-roofed shelter. Rectangular houses of stone with flat roofs have long been built by the Hopi Indians of Arizona.

5. *The square house*. Certain tribes, like the Penobscot Indians in Maine, preferred a *square* ground plan for their shelters. The roof of such a shelter was likely to be a pointed one. Each of the four sloping surfaces or "faces" of such a roof had the form of a *triangle*. The entire roof had the form of a *square pyramid*.

Modern buildings; the rectangle. The story of the early shelters readily suggests how the form of a modern house came about. Most of our homes now have a gable roof which rests on a rectangular foundation. On the other hand, our big office buildings, business blocks, "skyscrapers," and manufacturing plants, usually have flat roofs. Nearly all these buildings also have a rectangular ground plan. Their walls are rectangular, as well as their floors, windows, and doors. The boards in the floors, the bricks and stones, are rectangular. Naturally, building lots, as a rule, are also rectangular. This has led to straight streets which usually cut each other at right angles. The sidewalks consist of rectangular sections. Moreover, much of the furniture and other equipment in modern buildings is of rectangular form.

Hence it is that, as someone has said, "ours is a rectangular civilization." *The rectangle is easily the most practical of all the forms we study in geometry.*

Dominant forms. An analysis of the geometric forms arising in connection with such everyday practical activities as building, weaving, and farming, and in the industrial arts, shows that among

the plane figures used constantly, the *rectangle*, the *circle*, and the *triangle* are easily the outstanding forms. Moreover, the most common space forms are the *rectangular solid*, the *prism*, the *three round bodies*, and the *pyramid*. In other words, these forms seem to occur more frequently in the practical work of the world than any others. In particular, when we understand the *evolution* of these forms and the reasons for their constant use in everyday life, we are no longer in doubt about the reasons for including them in any truly motivated course of study in geometry.

B. THE STUDY OF SIZE

The evolution of measurement; farming as a source of geometric information. The most important of all human activities have always centered around the securing of an adequate supply of *food*. As soon as the ordinary stocks of food in any particular district became limited, it was necessary for people either to migrate or to think of some way of increasing their supplies. This accounts for the tilling of the soil, or *farming*. Moreover, it became necessary to *store* food and water. This gave rise to basketry and the making of pottery. Finally, food had to be *transported* from place to place, which caused the invention of the *wheel*. The exchange of commodities gradually led to barter and to other commercial enterprises. Storehouses and granaries were eventually constructed. All these and many other activities contributed to the development of geometry, especially to a knowledge of *mensuration*.

It has been known since the days of ancient Egypt that farming led to surveying, and thus caused the discovery of many geometric relationships. In fact, *geometry* means "*earth measurement*." Herodotus tells us that Egypt was the home of geometry. This was due to the annual floods caused by the Nile. These floods regularly destroyed many boundaries, thus making it necessary each year to survey the farms anew and to enter on the assessment rolls the exact area of each piece of land. In this way the basic techniques and rules of mensuration were discovered. Among the earliest of these was the rule for finding the area of a rectangular piece of land. The Egyptians also had rules for finding the area of a triangle and the area of a circle.[30]

[30] See R. C. Archibald's bibliography on Egyptian and Babylonian mathematics in the American edition of the *Rhind Mathematical Papyrus*, 2 vols., Oberlin, Ohio, 1927-1929.

It is probable, likewise, that at a very early time people learned to estimate the capacity of bins and granaries. In other words, farming and the related arts definitely established and developed a knowledge of the simplest processes of mensuration. The story of the early units and tools of measurement is of absorbing interest even to very young pupils.

Measurement as the master art. Just as the study of shape can always be motivated by reference to nature and to the practical arts, so measurement is indissolubly connected with the very warp and woof of our existence. It can be shown that modern life depends on accurate measurement at almost every turn. No one has explained this more clearly than Mr. Henry D. Hubbard of the National Bureau of Standards in Washington. To quote:

> *Measures rule the world.* All things are made to measure. To industry, measurement is the tool of creation. Industry is science set to measure. *Measurement is a miracle worker.*
>
> Here is a shoe factory with a houseful of lasts. The shoe sums centuries of shoemaking art in a set of measures—length, width, instep, ankle—by which the shoe is built, classified, sold, and worn.
>
> A dress pattern sums up an age-old art—clothing the body. Artists of the mode build their creations on measures of the body. Every cut of the shears or stitch of the needle is measured, to ensure perfect fitting for comfort, taste, or health.
>
> What is true of the shoe last and the dress pattern is true of a hundred thousand products of industry. If we need a hat, gloves, collars, the first question is *size*. The machine knows only the size, the user only the quality.
>
> *Modern science began with measurement. Measures are great teachers. They teach truth.* By modern methods, scientists measure accurately how things happen in nature and in experiment. Such measures build civilization. They are the numbers which rule the world of enterprise, the unseen frame of all achievement.
>
> The engineer puts measures to work in skyscraper, bridge, and other structures. Measures flow through his pencil to scale drawings. *By means of measures the engineer builds his dream of beauty.* When the cathedral stands finished, strong and beautiful, we forget the measures, but they remain forever the strength and beauty of the cathedral.
>
> *The blue print speaks the language of measurement.* In study, shop, and laboratory, the world of "Tomorrow" is being traced on paper with the aid of measured scales. *Measurements are thus shaping the pattern of the "Wonderlands of Tomorrow."* [91]

[91] For a more extended quotation of Hubbard's views on measurement, see *The Third Yearbook, The National Council of Teachers of Mathematics*, 1928, pp. 151-154.

C. THE STUDY OF POSITION

Nature's basic positions. The trunk of an upright tree is in a *vertical* position. Some of the branches are *horizontal*, or nearly so, while others are drooping or *slanting*. Again, the trunk of a tree and its shadow form a *right angle*. It is easy to see that these relations were bound to appear constantly in the products of the manual arts. Thus, the framework of a modern building is a configuration composed of perpendicular, parallel, and oblique lines Evidently this fact, as in the case of a tree, is due to the force of gravitation. Hence *perpendicularity* and *parallelism* are seen to have a background as lasting as the laws of nature.

The omnipresent rôle of position. Even a child learns very soon, in his home, that the desks, chairs, pictures, rugs, and all the other objects in the house, have definite places or positions. At a later stage, he finds out that, likewise, constant attention is given to the exact location of things in the school, in shops, in stores, and in offices. He is told that this definite arrangement of things is necessary for the sake of attractive *appearance* and for the sake of *order*. It has been said that "heaven's first law is that of *order*."

Again, we may think of a machine such as the engine in an automobile. We know that unless all its many parts are put exactly in the right place or position, the machine will not *work* and the car will not run.

When we study a city map, we see that it shows the location of all the important public buildings, such as schools, churches, fire houses, banks, and hotels. In the same way, the map used by a motorist shows the location of towns and cities, of roads, railways, and the like.

When we go to school in the morning, we find out the correct time by observing the position of the hands of a clock. Most of us do not realize that the stars, as they rise and set from day to day throughout the year, have enabled the astronomers to develop the *calendar*. That is, the positions of the stars represent a great clock by which we *keep time*. By observing the positions of the sun, the moon, and the stars, and by using his maps and instruments, a captain can steer his ship across the trackless ocean.

In countless trades and industries, practical people like builders, contractors, carpenters, and machinists depend on charts and blue prints which show every detail of the work to be done.

Every hour of the day, whether we are walking or resting, the correct position of our body is of importance to our health.

These and many other illustrations that might easily be given will serve to explain why much attention should be paid to a study of "position."

Symmetry and position. In creating the world of organisms, nature evolved an unending variety of forms. Many of these, such as the human body are *symmetric*. The *causes* of this symmetric arrangement are profoundly interesting, but cannot be discussed in this chapter.[92] It is sufficient to say that, for similar reasons, man was also led, unconsciously at first, to give due attention to the symmetric appearance of his manual and industrial products. In countless ways, symmetry accompanies us in every walk of life. A fruitful and simple field which illustrates the laws of symmetry is that of elementary *design*. It is not an accident that since the days of Dürer in Germany and Bouelle in France the study of symmetric figures has been suggested persistently as one of the most profitable aspects of everyday geometry.

Number and form united. The fact that numbers may be used not only to indicate *size*, but also *position*, is of the very greatest importance. Everybody is familiar with this idea. It is only necessary to recall the appearance of numbers on the dial of a clock and on measuring instruments such as rulers, thermometers, gas meters, and so on. The orderly position of these numbers underlies the reading and recording of definite measurements.

In the same way, by using number-pairs, we can locate points on a *surface* with the aid of *two* axes of reference. This is what happens whenever we locate the points of a graph with the aid of a table of values. The startling consequences of this simple idea, as suggested by Descartes and his followers, revolutionized mathematics.[93] It was Descartes who "presided over the marriage ceremony of algebra and geometry," by making possible the simultaneous or unified study of number and form.

[92] See Wheeler, R. H. and Perkins, F. T., *op. cit.* [4], pp. 392-396; Wheeler, R. H., *The Laws of Human Nature*, pp. 92, 134. D. Appleton and Company, New York, 1932; for a further mention of Dürer's symmetry of design, see page 220.

[93] The use of coördinates, in the modern sense, can be traced back at least as far as the *Tractatus de latitudinibus formarum* of the French clergyman and scholar Nicole Oresme (died 1382). An even earlier appearance of this idea was discovered in a manuscript of the tenth century. See Smith, D. E., *History of Mathematics*, Vol. II, pp. 319-320. Ginn and Company, Boston, 1923.

This notion of coördinates underlies navigation and surveying, astronomy and higher geometry. It is primarily responsible for the designation of mathematics as the "science of position" or the "science of order."

D. BASIC GEOMETRIC IDEAS

Congruence and similarity. The vast practical importance of congruence and similarity is not realized by teachers and pupils when the classroom work is limited to the usual "laws" relating to triangles only. And yet these ideas are illustrated in every direction by the products of our machine age. A Ford car is made possible only by the "mass production" that is characteristic of a modern factory. Millions of "parts" are turned out, as nearly alike as human ingenuity can make them. The same idea is used whenever designs, patterns, pictures, and plans are "duplicated." Any "reprint" is a duplicate of some original. Shoes made on the same last and according to the same style may be regarded as congruent. Castings coming from the same mold are congruent. Any two objects constructed in accordance with the same specifications illustrate the principle of congruence.

Whenever the scale ratio used in copying a plan differs from unity, we obtain a set of *similar* figures. Blue prints, enlarged or reduced photographs, maps, and even "moving pictures," are based on this idea. The industries often manufacture several "sizes" of an article, all built according to the same model or style, and hence similar.

Nature is full of suggestions of congruence and similarity. The leaves of any plant or tree, as well as animals of the same species, resemble each other. Crystals, fruits and flowers, the decorative patterns of organisms, the cellular structures of plants and animals, and the rhythmic motions of electrons, no less than those of the planets and the giant suns, proclaim the dominance of this idea. It is this very regularity of nature's patterns and motions which alone makes science possible.

In the light of the vast ramifications of these ideas, so completely ignored in the usual course in "demonstrative" geometry, should not a first course in geometry point out some of the romance that they suggest?

Symmetry. Axial and central symmetry, likewise, are of universal interest, as was pointed out above. Natural and manu-

factured objects illustrate this interesting relation in an almost endless variety.

In geometry there has been a growing conviction that the use of symmetry as a fundamental principle, through the idea of rotation and folding, is the best device we have for making many geometric properties clear intuitively. Unfortunately, our conservative high school geometry has avoided the extensive employment of symmetry as a tool, although a thoroughly rigorous treatment is possible on this basis,[94] and although great mathematicians have long given enthusiastic support to this mode of approach.

Equality, inequality, and proportion. A very large part of geometry centers around the idea of *equality*. In the case of congruent figures or objects, the corresponding parts are equal. We are surrounded on all sides by equal lines and equal angles. When figures are *similar*, however, the corresponding lines are *unequal*. Instead, there is now an *equality of ratios* of corresponding lines. We are thus led at once to the concept of a *proportion*. Ratio and proportion are therefore seen to be dominant relations in any course concerned with everyday geometry.

Functionality and variation. The idea of functional changes, long ignored in the average mathematical classroom, is now generally regarded as one of the corner stones of the mathematical edifice. To quote:

> Everyone knows that among the most impressive facts of our world is the great fact of *Change*. The universe of events, whether *great* or *small*, whether mental or physical, is an endlessly flowing stream. Transformation, slow or swift, visible or invisible, is perpetual on every hand. . . . To discover the laws of change is the aim of science. In this enterprise of science the ideal prototype is mathematics, for mathematics consists mainly in the study of functions, and the study of functions is the study of the ways in which changes in one or more things produce changes in others.[95]

The world of forms is a continuous demonstration of these trenchant words of Professor Keyser. When a stone is dropped into a pool of water, we soon observe a series of concentric circles traveling rapidly away from the center of disturbance. Any growing organism presents a functional picture. Its parts vary from moment to moment in accordance with the laws of its growth. The leaves

[94] See Fladt, K., *Elementargeometrie*, Part II, pp. 9-14, Leipzig, 1928.

[95] See *The Sixth Yearbook, The National Council of Teachers of Mathematics*, 1931, pp. 50-51.

of a tree exhibit numerous variables all tied together in one configuration, or pattern, or functional relationship.

In the same manner, we may view any geometric figure as a momentary, static member of a functional series. Thus, any given square is one of an infinite number of possible squares. All of them reflect certain underlying properties which do not vary, but remain "invariant." The perimeter of any *one* of these squares is governed by the formula $p = 4s$, and its area by the formula $A = s^2$. In each one, the diagonals are equal, the mid lines are equal, there are four axes of symmetry, and so on. By investigating the invariant properties or laws of all squares, we have gained control over the possibilities of any given square.

This is a simple and yet very profound thought, which can be developed profitably even in the most elementary course in geometry. Every geometric *formula* expresses an invariant property of a certain configuration. Every *construction* line is built into a functional background.

A wealth of construction problems can be devised along these lines. The pupils can be told to draw any three lines so as to form a triangle, and to measure the angles of the triangle. The sum of these angles is found to be always the same, a surprising and thrilling discovery to beginners! They can draw a set of concentric circles. In *each* case the circumference is found to be π times the diameter. Let the vertex of any triangle move along a path at a fixed distance from the base, and the area of each resulting triangle is always seen to be the same. Numerous other questions can be brought up, such as: What happens when the altitude of a triangle is made twice as large while its base remains the same? How does a plan change when its scale ratio is doubled? What is the effect on a solid when all its dimensions are enlarged or reduced?

But a chain of similar figures suggests not merely a set of invariant properties applying to any *one* of them. By *comparing any two members of the series* (e.g. two similar triangles) we obtain a set of *invariant ratios*. Thus we are led to the study of *numerical trigonometry* as a natural outgrowth of a general principle.

It is possible to view almost every classroom activity in mathematics in the light of functionality. As a unifying principle it is unexcelled, not merely in algebra, but also in geometry. "Functional thinking in geometric form" should become a cardinal part of our mathematical creed.

II. The Basic Instructional Materials of Intuitive Geometry

No agreement on organizing principles. The struggle for a universally recognized content and an established sequence of topics has been going on for many decades in the field of intuitive geometry. Yet, even if a second "Euclid" could offer a generally acceptable system, it is doubtful whether such uniformity would be desirable or necessary. Once we give up a strictly deductive type of organization, we have considerable liberty of action. But whatever one's organizing principles may be, there is still need of agreement on the main ingredients of the course. Again, once these are chosen, it will make a difference whether they are assembled in haphazard fashion or are made to fit into an organic whole that is both significant and appealing. In the following pages some of the basic geometric materials of instruction will be analyzed with a view to finding criteria for the evaluation of existing or proposed curricula.

A. Geometric Concepts

A basic list. We have attempted to trace in some detail the permanent natural setting of the fundamental geometric concepts. When this background is more fully appreciated, there will be less uncertainty about the selection and the relative importance of these concepts. A basic list should include terms such as the following:

Rectangle.	Parallel lines.	Congruence.
Square.	Perpendicular lines.	Similarity.
Circle.	Length.	Symmetry.
Triangle.	Width.	Equality.
Rectangular solid.	Height.	Line.
Cube.	Dimensions.	Angle.
Cylinder.	Area.	Measurement.
Prism.	Volume.	Ratio.

A considerable number of geometric terms might be added. The *New York State Tentative Syllabus in Junior High School Mathematics* lists more than 120 separate geometric concepts. Other published lists have helped to clarify the issue of determining the active vocabulary of intuitive geometry.[96]

[96] See Schorling, R., *A Tentative List of Objectives in the Teaching of Junior High School Mathematics*, pp. 101 ff., George Wahr, Ann Arbor, Mich., 1925; also, Smith, D. E. and Reeve, W. D., *The Teaching of Junior High School Mathematics*, pp. 44 ff., Ginn and Company, Boston, 1927.

Fallacy of frequency considerations. Curriculum investigations have often made the mistake of appraising the relative importance of mathematical terms by a count of the number of times such terms are used in magazines, books for the general reader, textbooks, and the like. Valuable as such studies have been, they certainly must not be taken as final. For example, even if the notion of *congruence* were not explicitly mentioned in a million pages of ordinary reading, it would still be true that it constitutes one of the most essential ideas of geometry. And though the average citizen of our day might never have occasion to handle a *meter* stick, we should still be under obligation to discuss the most scientific system of measurement produced thus far. In other words, "social utility" is far from being an exclusive criterion in the building of a course of study.

B. GEOMETRIC RELATIONSHIPS

Metric relations. A very large part of geometry is concerned with the metric properties of the geometric figures. In fact, measurement, both direct and indirect, is one of the chief concerns of everyday mathematics. In the trades and industries, in shops and laboratories, it is perhaps the most important aspect of geometry. The metric relations of geometric figures range from the simplest to the most complex. Thus, all right angles are equal. Vertical angles are equal Parallel lines, triangles, parallelograms, circles, all suggest important metric relationships. The rules for determining areas and volumes are metric. With every angle we may associate a fixed set of ratios which are the basis of trigonometry. Geometry originated as the science of "earth measurement," surveying. To-day it furnishes the means of exploring a metric domain which includes not only the tiniest electron but also the spiral nebulæ in the ocean of infinite space.

Positional relations. Though the world of forms exhibits infinite variety, the geometer always visualizes and studies these forms with the aid of a few idealized "elements" called points, lines, and surfaces. By combining or arranging these in a systematic way, he creates, as by magic, an unending set of configurations such as angles, triangles, polygons, circles, higher curves, and the world of solids.

If *three* sticks are taken and the various ways in which they may be used in order to form significant geometric figures are investi-

gated, parallel lines and the whole domain of triangles are at once available. When *four* sticks are used in a plane surface, it is possible to explore the properties of the various quadrilaterals that can then be formed. Significant "positional" questions will readily suggest themselves, such as: How many different relative positions can a point occupy with reference to a circle? Which configurations arise when we have a circle and a line? A circle and two lines? A circle and three lines? Two circles?

This game works both ways. It is both synthetic and analytic. Not only can we "combine" the simple elements into ever more complex structures, but we can break up a geometric "whole" into its constituent parts. Let it be recalled that it is this "combinatory analysis" which has given to the methods of geometry their vast potential power.

Functional relations. When we try to determine the properties of a figure that changes according to a fixed law, we are led to the idea of a functional relationship. Thus, if a point in a plane moves in such a way that it is always at a fixed distance from a given point, it describes a *circle*. Descartes found a method of stating such relationships "analytically." In the case of the circle, we obtain the characteristic equation $x^2 + y^2 = r^2$, familiar to most high school pupils.

To what extent should functional thinking of this sort be cultivated in elementary geometry? We should certainly refrain from troubling a beginner with the definition of a "locus." Nor is such a definition necessary. Countless "motions under constraint" illustrate the idea in which we are interested at this point. Can we not depend on simple questions similar to those in the following list:

1. Where are all the houses that are located one mile from your school building?

2. What is the path described by the door knob as you open the door? By one corner of the cover of your book as you open it?

3. When you open the drawer of a desk, what is the path of one of its moving corners? Of its edges?

4. When a street car travels along the tracks, what is the path of the center of one of the wheels?

A study of machinery in action can be made to suggest functional ideas to any pupil of average intelligence. For more ambitious minds, it is perfectly possible to include a few of the higher

plane curves. Thus, as a wheel travels along the ground, what is the path described by a point on its rim? With the aid of squared paper, or with the technique used by Mrs. Edith L. Somervell in the case of kindergarten children, a surprising amount of valuable functional work is possible.

C. GEOMETRIC FACTS

Number of facts unlimited. Obviously there can be no limit to the discovery of geometric truths. Also, there is no reason why at least the major propositions usually stressed in high school geometry should not be considered in an elementary course, even though they cannot ordinarily be proved *deductively* at that stage. All these geometric truths should, of course, be derived naturally in connection with the organic development of the subject. The selection of geometric facts in any classroom is largely guided by time allotment and the background to be created. It is desirable to attempt to classify these facts according to the scheme suggested for the study of relationships. Following this plan, we may speak of metric facts, positional facts, and functional facts.

Metric facts. In this group belong all the mensurational properties of figures. Here is a partial list of such propositions:

1. Vertical angles are equal.
2. The sum of the angles of a triangle is 180°.
3. The base angles of an isosceles triangle are equal.
4. The opposite sides of a parallelogram are equal.
5. The diagonals of a parallelogram bisect each other.

There are many such statements. Obviously, all the formulas concerned with areas and volumes should be included under this heading. Above all, the Pythagorean proposition is a fundamental metric fact of geometry.

Positional facts. In this group we have all the facts that pertain to the intersection of figures, the construction of figures, the uniqueness of constructions, and the like. This list includes statements such as the following:

1. Two straight lines can intersect each other at one point only.
2. Two circles which intersect each other have two and only two points in common.
3. If two isosceles triangles are constructed on a common base, the line passing through their vertices is the perpendicular bisector of the common base.

4. Only one perpendicular can be drawn to a given line through a given point.

Obviously, many propositions may be classified as both metric and positional. This is true, for example, of the usual laws concerning congruent and similar triangles, and of symmetric relations.

Functional facts. 1. Any *mensurational situation* which involves two or more variable elements is a fruitful field for functional considerations. Thus, in a formula such as $A = lw$, we may ask what happens to the value of A when l is doubled while w remains constant. In doing this kind of work, however, the attack should be purely pictorial for a long time. That is, *the relationships should be studied by means of diagrams rather than algebraic symbols.* A merely symbolic study of variation does not interest the beginner, nor does it convince him.

2. Here again, we have all the facts pertaining to "loci." Many of the truths that cause such difficulty in demonstrative geometry can be dramatized and made very real and meaningful even in the most elementary introductory course.

3. All *construction work* offers a natural background for "functional" questions on the part of the teacher Thus: draw a set of squares, of sides 1 in., 2 in., 3 in., and so on. Compare their perimeters and their areas. Can any three sticks be hinged together so as to form a triangle? If four sticks are hinged together so as to form a quadrilateral, investigate the limiting values of the diagonals. Consider a set of cubes having as edges 1 in., 2 in., 3 in., and so on. Study their surfaces. What do you observe? Can this conclusion be generalized? What is the practical value of this discovery?

Needless to say, at this point we have the most wonderful opportunity of correlating arithmetic, geometry, and algebra.

D. GEOMETRIC ABILITIES AND TECHNIQUES

A word of warning. The practice of dissecting any subject into its primary "bonds," skills, abilities, and other "objectives," and then subjecting these simple elements to the usual routine of piecemeal drill and testing, has been shown by the newer psychology of learning to be absolutely false. The incessant *repetition* of such "elements" does *not guarantee* learning. Without adequate, organic *motivation* that leads to *insight*, and without *the will to learn*, there is no real learning. Then, too, it cannot be stressed too often that in the field of geometry we cannot expect to

"get rich quickly." A proper time allotment must be provided for growth or "maturation." A few lessons in "constructive" or "experimental" geometry at the *end* of the school year, "if time remains over from the arithmetic work," will and should prove disappointing. "We never reap more than we sow." The principal reason why geometry so often fails in our schools, aside from the absence of genuine motivation and good teaching, is the haphazard introduction and hasty presentation of the geometric materials.

The use of the tools. Each instrument presupposes a "mastery technique." A carpenter becomes efficient in using the tools of his trade by constantly employing them in meaningful situations. In learning his trade, he does not drive nails for two weeks, saw boards for a month, use the chisel or the screw driver alone for ten days, and so on. In the same way, pupils do not become expert in using geometric tools by a few specific drill lessons. To be sure, the technique of handling each instrument must be clearly explained, rehearsed, and illustrated, to avoid waste of time and materials. After that, the continuous, organized use of these instruments in significant geometric situations will gradually lead to a really proficient response on the part of the pupil.

The standard equipment for work in intuitive geometry includes the ruler (English and metric scales), the compasses, the protractor, and a supply of squared paper (both English and metric units being desirable). A more ambitious course also calls for the customary instruments used in mechanical drawing, and for simple surveying instruments. Much of this auxiliary equipment can, of course, be made in the school shops. The course of study must be planned in such a way that these instruments are given adequate attention *throughout the school year.*

Classification of geometric abilities. If the geometric classroom activities are conducted in the proper manner, and if a sufficient amount of time is provided, the pupils will acquire more or less completely all the desirable abilities that have been enumerated in recent curriculum studies and in the current literature on junior high school mathematics. These abilities may be classified as (1) mechanical, (2) conceptual, (3) practical, and (4) appreciational. Thus, the pupil will learn to identify and classify geometric forms. He will eventually be able to draw or construct the fundamental geometric figures. He will know how to measure line segments and angles within a reasonable margin of error. He will

find it possible to visualize and to imagine geometric figures, including three-dimensional figures. He will desire to carry on simple geometric experiments leading to new discoveries. He will gain confidence and skill in applying the principles and rules he has learned in connection with everyday life problems. Finally, he will "experience" geometry in his surroundings and will appreciate its all-embracing atmosphere.

E. GEOMETRIC DRAWING AND DESIGN

General importance of this field. It has been said that "the fine arts are the best reflection of civilization." Many of the fine arts have a geometric foundation. This is true of such "graphic" arts as designing, printing and lithography, painting and etching, of sculpture, and of the making of pottery and jewelry, of fine metal work and the use of art in advertising. Numerous industries, such as those concerned with the making of clothing, furniture, and textiles, with rugs, linoleum, wall paper, metal work and plumbing, as well as the practical activities of the architect, the interior decorator, the engineer, and the surveyor, would be crippled without a knowledge of the geometric methods employed in drawing and design. The blue print is the language of the trades, the map is the passport of commerce and travel, the graph is the barometer of industry, and the photograph—nature's automatic drawing—has become the indispensable instrument for bringing entertainment to the masses.

In an address delivered several years ago by Dr. Thomas J. McCormack, this inspiring passage occurs:

> Reality is what *ought* to be, not what is. . . . The life of sense is the life of the animal, with all its immediacy; incoherent, unorganized, chaotic, with no before or after. *The life of man is the life of order, the life of control, the life of purpose, the life of the idea and the ideal,* born of the past and avid of the future. All other life is unhuman. *Where the idea is lacking, there art, and science, and philosophy are lacking, and existence sinks to the animal level. The characteristic of man is thought. The machinery of thought is the idea, the universal. . . . The vestment of thought is language. Philosophy, art, science, literature, all are simply species of one great genus—expression. And expression is the daughter of reason, not of sense.* "Heard melodies are sweet, but those unheard are sweeter." Let Phidias carve and Sappho sing; but fairer than the fairest form ever carved, fairer than the most exquisite musical note ever uttered is *the rapturous vision of Beauty herself,* clear and unalloyed, absolute, simple and everlasting. Approximation to this ideal in conduct and expression, and not the

ravishment of the senses, is the really human life, the only life worth living. *What ought to be always transcends what is. It is the standard and touchstone by which we test our loyalty to our higher selves. In every human life, however low, the ideal is triumphant, some great desire transfigures the inward existence. We never attain the vision, the dream which we set, but the dream is always cherished as the ultimate reality.*

The fact is that the great and the most beautiful part of life is vicarious. We live largely by proxy, take our sunshine from others, and suffer, not one, but a thousand luminaries to irradiate us. *We are moons, not stars. This is the great economy of literature, art and science.* In no other way can we compass life and the universe. *They abbreviate infinity for us, and give us immortality in the passing moment.*

The thing that haunted us for years, that begged for utterance for generations, appears, intelligible and glorious. *It is discovery, revelation.* It is exactly what Kepler did when he unriddled the complexity of the heavens, or Galileo when he untangled the skeins of earthly motion, or Darwin when he imprisoned in a word the process of unfolding life. *We see it when the forms of Rodin spring from the shapeless marble. It is the epiphany of the idea—the same in every field of human thought and expression.*[97]

Classroom possibilities. A great gold mine of valuable geometric training and appreciation lies imbedded in the field suggested above. So far, only a beginning has been made in bringing these treasures into the schoolroom. A considerable body of literature on this subject is now available, both here and abroad. Our very limited time schedules do not permit a very liberal margin for this sort of work. And yet, any teacher who has observed the genuine joy and enthusiasm with which geometry pupils respond to every opportunity for free "expression work" of the artistic type, will try to "create time" for this appealing and arresting form of classroom activity. In any case, it is always possible to depend on the stimulus of "supplementary honor work," of exhibits and source book displays, and thus to find an outlet for this phase of applied geometry.

It was pointed out above that Pestalozzi had a very high regard for geometric drawing. The Herbartians developed this field to a considerable extent. Thus, Dr. E. Wilk presented a valuable summary of the origin and the possibilities of *historic ornament*. We have also called attention to Fröbel's "forms of beauty."

In many of the recent European curricula, geometric drawing has become an integral part of the work in geometry, extending

[97] McCormack, Thomas J., *Beyond Good and Evil*, pp. 6-7, address given before the English section of the State High School Conference, University of Illinois, Champaign, Ill., November 23, 1923.

over a period of several years. What has already been accomplished in this direction may be inferred from the writings of Jahne and Barbisch in Austria, of Becker, Treutlein, Lietzmann,[98] Wolff,[99] Schudeisky and Ebner [100] in Germany, of Morris, Spanton, and Mrs. Somervell in England, of Bourlet [101] and Baudoin in France, and of Garbieri [102] in Italy.

The National Committee of Fifteen on Geometry referred to some of the literature on applied geometry, including the field of design. Miss Mabel Sykes of Chicago did pioneer work which is of permanent value.[103] The publications of our large museums are full of excellent material. Trade catalogues, books on applied art, periodicals and advertisements, and many similar sources, will increasingly assist in transforming the geometry of the classroom into a living, enjoyable reality.

PART FOUR

The Organization of the Instructional Materials

I. Possible Curriculum Approaches

A. A TOPICAL ORGANIZATION

The synthetic approach. This approach is the old "systematic" one which, until quite recently, held complete sway in every school subject. In geometry, it began with points and lines, and proceeded step by step "from the simple to the complex"—a Pestalozzian idea fostered and abetted by the old association psychology. For an *adult* this is a very logical and satisfying scheme. It is out of place in the schoolroom because it fails to put "first things first," when such criteria as interest, social utility, and economy of learning are applied. Hence it is being abandoned in favor of other plans that provide both meaning and orientation with less effort and greater certainty.

The analytic approach. In this case the subject is organized around large categories, such as shape, size, and position. All minor

[98] Lietzmann, W., *Matnematik und bildende Kunst*, Breslau, 1931.

[99] Wolff, Georg, *Mathematik und Malerei*, Leipzig, 1925.

[100] See Ebner, Max, *Ausführliche Stoffauswahl für die Lehrpläne im wissenschaftlichen Zeichnen*, Leipzig, 1924.

[101] Bourlet, C., *Cours abrégé de géométrie*, Paris, 1907.

[102] Garbieri, Giovanni, *Geometria intuitiva e disegno geometrico*, Paravia and Company, 1904.

[103] Sykes, Mabel, *Problems for Geometry*, Allyn and Bacon, Boston, 1912.

topics or activities are then brought into relation to these broad divisions. This mode of approach integrates the work under inclusive headings that give meaning to the subsidiary items. In other words, the point of view is organic and "configurational." It is more natural than the synthetic plan and it is more appealing to the learner. Its danger lies in an overdose of certain fixed materials at one time, thus leading to monotony and weariness.

The following list shows a topical organization of the sort suggested above:

1. *Shape*
   a. Definitions (vocabulary).
   b. Notation (symbolism).
   c. Classification of forms.
   d. Properties of figures (facts and relationships—informational).
2. *Size*
   a. Measurement (direct and indirect).
   b. Formulas (areas and volumes).
   c. Relations (metric and functional).
3. *Position*
   a. Constructions.
   b. Drawing and design.
   c. Facts and relationships (positional and functional).

B. A PROJECT APPROACH

Advantages. The use of genuine "life situations" in the school has a direct motivating power that no other plan of teaching can possibly surpass. This approach would, most certainly, largely eliminate the perennial accusation that the conventional school is merely a *preparation for life, not life itself* (Dewey). And so, our textbooks are now making desperate efforts to do justice both to this ideal of immediate interest and to "life participation."

Difficulties. The proud boast of the project enthusiasts is that at last they have found a way of "bringing life into the classroom," that children originate their own curriculum materials, that there is spontaneity and self-activity where previously one could find only imitation and mechanical drill. But no real geometric project curriculum seems to have been developed so far. The attempt of Martin and Schmidt in Germany (see page 80) appears to have been unique. The reasons are obvious. Among the outstanding defects of a project organization are the following:

1. Real projects cannot be codified in book form.

2. Projects endanger or make impossible a systematic development of the basic objectives.

3. In an efficient school system the mastery of the essentials cannot be left to the chance inspiration of teachers and pupils.

4. Slow pupils cannot integrate or assimilate the mary "incidental learnings" suggested by a real project.

5. In the case of parallel classes in the same school, the administration of divergent project plans is virtually impossible, and uniformity is the one thing proscribed in a project curriculum.

6. A consummate amount of pedagogic skill is required, such as many teachers do not possess, to bring out and to organize the educational "lessons" of a significant project.

C. A "UNITARY" ORGANIZATION

Meaning and possibilities. A "unitary" organization, as advocated and popularized by Professor Henry C. Morrison [104] and his followers, is the combined product of a Herbartian teaching technique and the tenets of the measurement movement. Hence its good features and its obvious flaws. The idea of segregating the instructional materials of each school year into a moderate number of "units," each having its own "central theme" which is rehearsed until a "mastery test" reveals satisfactory scores, is a simple and alluring one. Insofar as a "unitary" organization operates in terms of large, inclusive topics or problems, it gets away from trivial "elements" and emphasizes *organic* learning. This is in harmony with sound psychological doctrine. Besides, the plan certainly tends to cause definiteness and administrative efficiency. It controls the pupil's growth as does no other plan. Many teachers have come to welcome it.

Limitations. Among the dangers and misconceptions which are certainly invited by a unitary organization are the following:

1. Whether a given piece of school work can or should be mastered "once and for all" by a continuous exposure to that phase of the subject is very doubtful. This idea of emphasizing one topic at a time to the point of mastery requires close scrutiny. The newer psychology has shown the *importance* of "spaced learning," of *repetition after adequate intervals*, and of *provision for maturation.*

[104] Morrison, Henry C., *The Practice of Teaching in the Secondary School*, University of Chicago Press, Chicago, Ill.. 1926.

2. A unit which is too extensive causes monotony and aversion.
3. A comprehensive unit may involve too many objectives, with the resulting danger of confusion.
4. A unit organization often demands much supplementary work on the part of teachers and pupils.
5. The organic interweaving of relatively independent units is not an easy task.

D. A PSYCHOLOGICAL VIEW OF THE CURRICULUM

The curriculum from the standpoint of the learner. Since the days of Pestalozzi and Herbart, educators have tried increasingly to view school life through the eyes of the pupil. The school has tended to become "child-centered," often forgetting that the child must eventually face adult standards whether he likes it or not. Yet, an exclusive emphasis on the *materials* of instruction, at any age level, without due regard for their probable influence on the immediate emotional and conceptional life of the child, is certainly no longer defended by anyone. And so, we may view the curriculum as a many-colored spectrum whose primary colors relate to such ingredients as *knowledge, skills, habits, attitudes*, and *types of thinking*.[103] Some of these c[illegible]ies obviously refer to subject

Objective		Subjective		
K	S	H	A	TH

matter and some to the learning process. It is a mistake, however, to treat these components in water-tight compartments. Thus, while attitudes may be *listed* in a printed course of study, they should really be thought of as part of the ever-present atmosphere or *spirit* of the classroom.

Importance of a psychological orientation. The new psychology has thrown a flood of light on the learning process. It has established the "organismic" point of view. "When we think, we think all over," says Herrick. The old "laws of learning," as proclaimed by the orthodox "bond" psychology, especially the *law of frequency* and the *law of effect*, are being rejected or revised in the light of recent, crucial experiments. Says Kilpatrick:

[103] Mead, A. R., *Learning and Teaching*, Chap. X, J. B. Lippincott Co., Philadelphia, 1923.

It appears that in each instance of behavior as above described the whole organism, and in some measure each constituent part, is involved. If this be so . . . . learning then takes on a very broad meaning. It becomes not simply the acquisition of a new way of behaving as commonly conceived. Rather does each new way of behavior mean in some degree a remaking of the whole organism. No part is omitted: intellectual insight . . . . emotional changes . . . . glandular readjustments, neuro-muscular readjustments. Some remaking and reorganizing is effected throughout, the degree of change at any point depending on the character of the experience as a whole and on the connection of the particular point with the rest. The far-reaching significance of this conception no intelligent theory of education can disregard. The whole organism is in some degree changed in each learning experience.[106]

Above all, attitudes, ideals, and types of thinking are now conceded to be the most potent carriers of "transfer." The long debate about "mental discipline" has resolved itself primarily into a question of *method*. The problem of "transfer," says Judd, is one of "working for transfer."[107] And, according to Wheeler, transfer involves an organismic and configurational background.[108] Hence, we cannot be too insistent on adequate means of motivation, on a continuous struggle for perspective and real mastery, and on a scrupulous regard for daily lesson plans that are both logical and psychological.

E. A COMPOSITE PLAN

Necessity of a compromise. Teaching is primarily an *art*, rather than a science. It will remain so until we have progressed far beyond our present degree of insight into mental and physical phenomena. And for that reason the school will continue to utilize and to conserve tested procedures that have been worked out through incessant labor by generations of teachers. The errors of each educational program or theory are at last made apparent, and a new and better synthesis is proclaimed on the banners of progress, only to be supplanted eventually by a still more promising reorientation. The story of curriculum revision is one of constant

[106] Kilpatrick, W. H., "A Reconstructed Theory of the Educative Process," *Teachers College Record*, Vol. 32, p. 534, March, 1931.

[107] See Betz, W., "The Transfer of Training, with Particular Reference to Geometry," *The Fifth Yearbook, The National Council of Teachers of Mathematics*, 1930, pp. 149-198.

[108] Wheeler, R. H., *Science of Psychology*, pp. 302 ff., Thomas Y. Crowell Company, New York, 1929; also Wheeler, R. H. and Perkins, F. T., *op. cit.* [4], pp. 318-327.

trial and error. The axiomatic formulations of each age are ridiculed by the on-coming reformers and iconoclasts of a new day, who in turn become obsessed by the fixity of their own visions.

In organizing a program for the teaching of intuitive geometry, may we not conserve the best features of all the plans sketched above? There is but one serious stumblingblock. It is the well-nigh universal, slavish cult of the textbook and of prescribed curricula, the comfortable but paralyzing conviction that a single printed document can impersonate or replace a living, enthusiastic, inspired teacher. Once this radical error is given up, the door is open for a flexible and ever-progressing type of curriculum.

Nature of compromise. A textbook must necessarily present a definite, systematic program. Every item must be listed on a specified page. Developments, exercises, definitions, summaries, tests must all be introduced in a more or less stereotyped fashion. But *the teacher is not bound by these codified materials.* The *interpretation* given to the instructional materials and the *life* that is infused into them, are the real test of good teaching. Two musicians may attempt to play the same composition. In one case we may turn away in despair, while in the other case we are entranced by the magic touch that gives a haunting beauty to the score.

In the same way, one teacher of geometry may decide to begin with a *project* that promises to arouse immediate interest. Another teacher may depend on a *motivating discussion* and accomplish the same purpose in less time. In one classroom we may find a rather unpromising *topical plan* which is yet made to yield good results by the pedagogic skill of the teacher. Another schoolroom may place the emphasis on a series of large "units" which are never allowed to become boring. In one community the teachers may be compelled to follow *uniform* lesson plans, methods, and time schedules, accompanied by standardized tests. Another school system may grant much greater freedom to its teachers without being necessarily superior.

Absolute uniformity of attack, of organization and method, presupposes a pedagogic Utopia. Too many variables are constantly at work. Nevertheless, the preceding pages have attempted to create an interpretative background in the light of which it will be less difficult to evaluate existing or proposed curricula, some of which we shall now examine.

II. The Curricula of Other Countries

A. THE TIME ELEMENT

The international situation. The graphic representation (on page 127), reproduced from Mr. J. C. Brown's monograph on mathematical curricula,[109] shows the number of years devoted to the study of geometry in thirteen of the foreign countries which are listed in that document. While many of these curricula have been modified in the meantime, the time allotments have not been changed sufficiently to rob Mr. Brown's summary of interest after a lapse of two decades. New studies are now being prepared and will become available in a few years.

It appears, then, that in all the leading foreign countries instruction in geometry is begun not later than the seventh school year, that it usually extends over a period of several years, and that in some cases the work is begun as early as the fourth or the fifth school year.

B. RECENT ENGLISH CURRICULA

Report on the Teaching of Geometry in Schools (1923). In this document, referred to above, the work in intuitive or "experimental" geometry is outlined as follows:

Stage A: The Experimental Stage. Here the boy will meet with the common geometrical notions and figures; a proceeding corresponding to the old preliminary course of definitions, but differing widely in this respect, that the treatment is no longer mainly verbal, but arises from real problems such as land measuring ("boy-scout geometry"), and is illustrated by the use of drawing instruments and other simple apparatus. As the course develops a certain number of fundamental facts emerge, relating to angles at a point, parallel lines, and congruent triangles. Opportunities for the introduction of deductive work will occur at this stage and should be utilized, the transition to Stage B being effected gradually. The treatment of deductive work in Stage A should be simple, and should in the main be conducted orally. Formal written proofs should be postponed to the next stage.

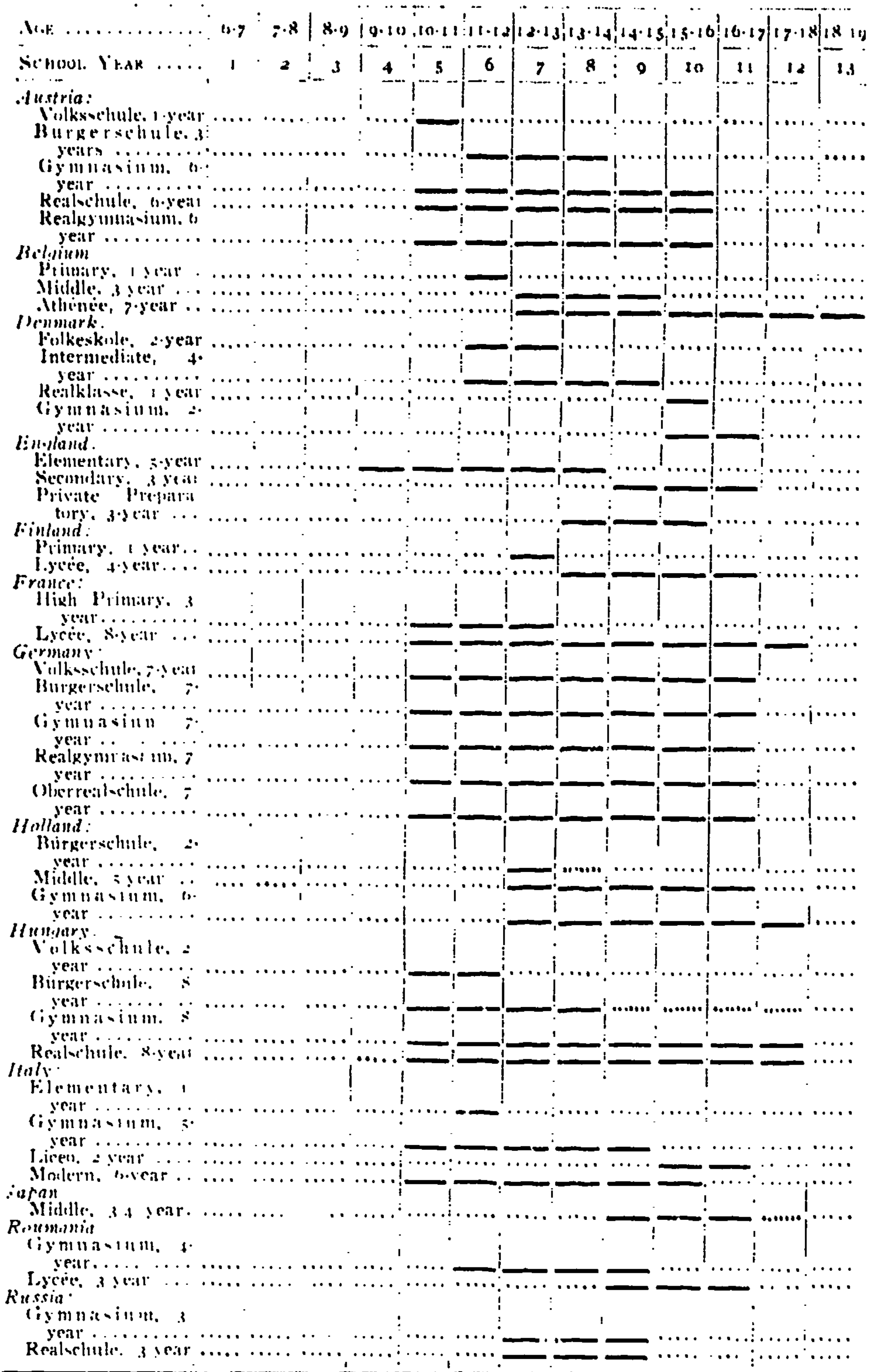
Stage A is essential as a preparation for Stage B, but unless it is arranged carefully there is a danger of its becoming desultory and aimless. It should end at the age of about 12½ years; thus an average boy who is on the way to a public school should have entered the next stage before he leaves his preparatory school.

. . . .

The first steps in geometrical work will be largely experimental, and will be connected with arithmetic and geography. Land measurement seems to

[109] Brown, J. C., *op. cit.* See [6].

YEARS OF STUDY OF GEOMETRY



form an important part of this work, and it is suggested that if circumstances allow, some exercises should be worked out of doors, not, it need hardly be said, forming a continuous course preliminary to all class-room work, but interpolated appropriately from time to time.

Apparatus. This should be as simple as possible. Expensive apparatus is not merely unnecessary but undesirable, both because the supply will generally be small and because elaborate details are apt to obscure the main principle.

. . . .

Playground or field work. This may include: measurement of direction round a fixed point, determination of position by direction and distance, planning to scale, measurement of height by set square and by angular elevation, determination of position and distance by triangulation from a fixed base, measurement of area by base-line and offsets, measurement of slope in various directions from a fixed point, and the plotting of sections and contours.

Class-room work. There is much room for variety in the order in which this is done; the course should include:

Measurement of length, work with decimal scale, estimation of fractions of a scale division, comparison of units (inch, centimetre, etc.); marking of points at a given distance from a given point, fixing of position by distances from known points.

Acquisition of the idea of a line as a boundary or edge, testing of the straightness of a line and of a ruler-edge; acquisition of the idea of a surface, testing of flatness.

Finding of areas (e.g., of regions on a map) by the counting of squares; discussion of area of a rectangle, and of volume of a rectangular block; construction of card models of simple solids.

Discussion of the rotation of a radius, measurement of angles; making of plans.

Discrimination of vertical lines and horizontal planes, of horizontal lines and vertical planes; study of the translation or sliding of figures.

Introduction to parallel lines; recognition of corresponding and alternate angles; development of the idea of direction; walking of boundaries; discovery of the angle-sum of triangle and polygon, construction of regular polygons.

Copying of a given triangle, construction of a triangle from data, introduction of the distinction between satisfactory and unsatisfactory (inadequate or incoherent) groups of data; recognition of congruence of triangles, and of similarity of triangles in the three-angle case.

The principal ruler and compass constructions (bisection of lines and angles, etc.), and appreciation of symmetry in the figures described.

Determination of areas of parallelogram, triangle, and trapezium, by dissection.

A few sentences from a later section on Stage B are added for the sake of completeness.

Stage B: Deductive. This stage is transitional, leading from the boy-scout geometry of Stage A to the formal geometry of Stage C. It is an essential feature of the Committee's recommendations that there should be such a transitional stage, that it should deal with matter selected from the substance of Euclid, Books I-IV and VI, together with some solid geometry, and that it should be an important part of the whole course, taking up perhaps 2½ years, say from 12½ to 15. But the very fact that the stage is to be transitional makes it difficult, and as yet undesirable, to frame very precise proposals.

The method is to be mainly deductive, but not without appeal to intuition.

Dr. Nunn's syllabus. In 1925 Dr. T. Percy Nunn issued *an Elementary School Syllabus in Mathematics*, which contains a continuous course in geometry beginning as early as the *second* school year. A synopsis of this syllabus may be found in a recent volume published by Dr. J. Shibli.[110] The earlier stages of this proposed course suggest numerous exercises in drawing, measuring, and surveying. The development is very gradual. Much attention is given to solids and to the basic geometric ideas (symmetry, congruence, similarity).

A number of English textbooks have appeared which are constructed along the lines of the syllabi mentioned above.

C. RECENT GERMAN CURRICULA

The secondary schools. The great political transformation of Germany has brought many changes into its educational system. As with us, curricular readjustments have ranged all the way from minor adaptations to a fundamental reorganization. Again, the new German *Reich* is still a federation of relatively antonomous states, as in the case of the United States, so that one will look in vain for absolute uniformity of educational programs and practices. Hence it is impossible to present a single comprehensive picture of recent developments. A fairly accurate impression is obtained only by inspecting the curricula of the major territorial units. The following account is an abstract based on the authoritative summaries of Dr. W. Lietzmann of Göttingen, whose encyclopedic grasp of the situation has made him the leading spokesman of the German reform movement, which is based in the main on the *Meran Report* of 1905 and its subsequent revisions.[111]

[110] Shibli, J., *Recent Developments in the Teaching of Geometry*, pp. 60-62, J. Shibli, State College, Pennsylvania, 1932.

[111] Lietzmann, W., *op. cit.* [23], Vol. I, pp. 225 ff.

Prussia (Regulations of 1925).

The cultivation of space intuition and the comprehension of the functional interdependence of variable elements in a given setting should be stressed at all times. Mathematical phenomena in the pupil's environment are to receive constant attention. Applied problems should be based on real data and on life situations. The pupil should be trained from the beginning in accurate oral expression and in a clear exposition of his knowledge.

The point of departure in geometry should be the study of simple solids. There should be much work in estimating and measuring, in the making of models, and in the use of the geometric instruments.

There should be a gradual transition from intuition to logical deduction. The idea of *motion* and of *variation*, as applied to plane and solid figures, is to be used extensively. Special emphasis should be placed on *axial and central symmetry*, on rotation, translation, and folding.

In addition to the ruler and the compasses, the instruments of mechanical drawing are to be introduced. From the very beginning *geometric drawing* is to be emphasized. A "considerable amount of time" is to be set aside for this purpose. The *relations of art to geometry*, including the significance of perspective, should be carefully illustrated in the classroom.

Measurement is to be one of the major objectives. The degree of accuracy is to be increased gradually by more refined methods.

The great importance of mathematics in the modern world should be discussed and explained continuously. Historical backgrounds are recommended as of extreme value.

Supplementary manual and shop activities are endorsed. The pupils should be encouraged to make models and to construct other illustrative devices. Under the general guidance of the teacher, mathematical club activities are to be offered, for a further study of supplementary topics of special interest.

This program is to be carried out, in connection with arithmetic, in the *first two years* of the secondary schools (ages 10 to 12). The fundamental idea, of course, is *preparation* for the work of the later years. The Prussian program is in close agreement with the general point of view of the Bavarian.

Bavaria (Regulations of 1914).

Functional thinking, space intuition, a clear grasp of mathematical ideas and principles, skill in solving applied problems, accurate verbal expression, avoidance of mere memorization, an intelligent reaction to mathematical situations suggested by the pupil's environment—are to be dominant aims throughout.

In geometry, practical illustrations based on the schoolroom and the school premises are to be the point of departure. The use of the geometric instruments is to be stressed. All drawings, at the board or in notebooks, are to be executed with great care.

This preliminary geometric work is to be done during the *third* year of the secondary schools (ages 12 to 13).

The official curriculum of Württemberg, though very similar to the two preceding programs, offers a few differentiating features.

Württemberg (Regulations of 1912).

At *all* stages, intuition is to be given the broadest possible scope. All memorization work is to be reduced to a minimum. Illustrative devices are to be used wherever possible. The manual activities of the pupils are to be encouraged. Historical considerations are to permeate the work of *all* classes. Correct oral expression and accurate, succinct presentation must be insisted upon. All written work must be arranged systematically and executed neatly. Geometric drawings are to exhibit an ever-increasing degree of accuracy.

In geometry, instruction should begin with the simple solids, from which the fundamental concepts, the positional relations of lines and planes, and the principal geometric figures are derived. "Scientific" definitions are to be avoided. The use of the geometric instruments (including those employed in mechanical drawing) gradually leads to the fundamental constructions. Through empiric methods (translation, rotation, folding, and measurement) the principal propositions concerning angle relations and concerning areas and circumferences are obtained.

The pupils are to have constantly at hand paper models of the principal plane figures, and to make cardboard models of the principal solids.

There is to be a gradual transition from intuition to demonstration. Even at a later stage, many obvious truths may be assumed intuitively.

From the outset, geometric figures should not be regarded as rigid. An extensive use of *motion* is recommended for the purpose of illustrating and suggesting important geometric relations. Three-dimensional exercises should be included whenever possible.

Geometric drawing is to receive attention throughout. Colored crayons should be employed extensively, and all blackboard drawings should be as accurate and attractive as possible.

As in Bavaria, this work is assigned to the *third* year of the secondary schools, along with the usual course in arithmetic.

The elementary schools. The official regulations of 1922 furnish only general directions, and fail to give detailed objectives for the work of each school year. An inspection of recent elementary textbooks, manuals, and courses of study leads substantially to the following picture of present-day German trends and demands in the field of intuitive geometry:

1. The cultural and æsthetic aims are beginning to receive as much attention as the strictly practical considerations.

2. Purely theoretic and abstract topics are being dropped in favor of everyday, worth-while materials of instruction.

3. The sequence of Euclid is rejected *in toto*.

4. The extreme project idea (Martin and Schmidt) together with the "incidental" discovery of geometric truths, is rejected.

5. The immediate environment of the pupil is used extensively as a geometric background.

6. Applied problems are stressed throughout.

7. Functional considerations and three-dimensional thinking are receiving due attention.

8. Tables and graphs are being used extensively.

9. Symmetry and motion are found to be of great value.

10. Mathematical amusements are introduced occasionally.

11. Life situations are being preferred to models.

12. Outdoor exercises (surveying) are meeting with favor.[112]

13. The geometric instruments, often including those used in mechanical drawing, are employed throughout

14. Mensuration is made real by the measurement of actual objects.

15. Models and other illustrative materials are made by the pupils themselves.

16. Socialized discussions, guided by the teacher, are replacing formal lessons and stereotyped recitations.[113]

17. Book problems are being subordinated to problems suggested by the pupils.

18. Mechanical procedures are being avoided.

As with us, the extremists are being held in check by experienced and conservative minds, and the ultimate goal is a better compromise between the old and the new.

Teichmann gives a complete course of study in geometry for the last four years (Grades 5 through 8) of the elementary school.[114]

E. Engel, an experienced Berlin educator, advocates three stages of geometric instruction, extending from the kindergarten to the last year of the elementary school.[115] The work of the *lowest* stage is based on ideas of Fröbel and Dr. Montessori. The *middle* stage comprises a study of directions, of measurement, of rectangles and

[112] Timmermann, H., ***Raumlehrestunden im Freien***, Union Deutsche Verlagsgesellschaft, undated.

[113] See, for example, Scharrelmann, H., *Produktive Geometrie*, Braunschweig, 1922. This publication reminds one strongly of George Iles' *A Class in Geometry*, published by E. L. Kellogg and Co., New York, 1914.

[114] Büttner, A., edited by Teichmann, O., *op. cit.* [23], pp. 40 ff.

[115] Engel, Ernst, *op. cit.* [23].

squares, rectangular solids and cubes, circles and cylinders. The *last* stage completes the usual program of the elementary school, including angles, triangles, quadrilaterals, polygons, circles, areas and volumes, congruence, similarity, symmetry, surveying, perspective, and geometric design. A vast amount of pedagogic experience is incorporated in this volume of 308 pages, which is perhaps the most original and up-to-date discussion of the past decade. Its translation would be a boon to American teachers. Its obvious flaw is a one-sided though masterly employment of *motion* as the *exclusive* generating and organizing principle of geometry.

Defects of German curricula and textbooks. At all times the Germans have been their own most severe educational critics. Long and stormy have been their debates about almost every imaginable school question. Mathematical reform has been no exception to this rule. Thus, it is a constant source of regret to many, as Dr. Lietzmann points out, that the elementary and secondary schools do not have a unified program for the introductory courses, a defect which is expected to continue until all the teachers of mathematics have enjoyed the same degree of academic training.

Among the shortcomings of the present courses in intuitive geometry the following are usually mentioned as particularly objectionable:

1. There is no recognized organizing principle.
2. The elementary schools still "ape" Euclidean traditions, with a view to creating a "scientific" impression.
3. The secondary schools are interested too exclusively in logical, systematic procedures.
4. Minor topics are often given an unwarranted emphasis.
5. There is still too much Pestalozzianism in the sequence of topics (e.g. beginning with the straight line or the cube).

III. Recent American Syllabi and Textbook Tendencies

Syllabus of the National Committee on Mathematical Requirements (1923). In this program intuitive geometry is treated mainly as a *preparatory* subject, in a 6–3–3 organization, precisely as in the secondary schools of Europe. In the section relating to intuitive geometry, in Chapter III of the Committee's report, the following topics are recommended:

Intuitive geometry. (*a*) The direct measurement of distances and angles by means of a linear scale and protractor. The approximate character of measurement. An understanding of what is meant by the degree of precision as expressed by the number of "significant" figures.

(*b*) Areas of the square, rectangle, parallelogram, triangle, and trapezoid; circumference and area of a circle; surfaces and volumes of solids of corresponding importance; the construction of the corresponding formulas.

(*c*) Practice in numerical computation with due regard to the number of figures used or retained.

(*d*) Indirect measurement by means of drawings to scale; use of square ruled paper.

(*e*) Geometry of appreciation; geometric forms in nature, architecture, manufacture, and industry.

(*f*) Simple geometric constructions with ruler and compasses, T-square, and triangle, such as that of the perpendicular bisector, the bisector of an angle, and parallel lines.

(*g*) Familiarity with such forms as the equilateral triangle, the 30°-60° right triangle, and the isosceles right triangle; symmetry; a knowledge of such facts as those concerning the sum of the angles of a triangle and the pythagorean relation; simple cases of geometric loci in the plane and in space.

(*h*) Informal introduction to the idea of similarity.

The work in intuitive geometry should make the pupil familiar with the elementary ideas concerning geometric forms in the plane and in space with respect to *shape, size, and position*. Much opportunity should be provided for exercising space perception and imagination. The simpler geometric *ideas and relations* in the plane may properly be extended to *three dimensions*. The work should, moreover, be carefully planned so *as to bring out geometric relations and logical connections*. Before the end of this intuitive work the pupil should have definitely begun *to make inferences and to draw valid conclusions from the relations discovered*. In other words, this informal work in geometry should be so organized as to make it a gradual approach to, and provide a foundation for, the subsequent work in demonstrative geometry.

In several respects this syllabus obviously goes beyond the traditional offerings. No time schedule is suggested, although each of the five model plans mentioned on pages 29-30 of the report assigns intuitive geometry to the seventh and eighth grades. Concerning schools still organized on the 8–4 plan, the following significant passage is added:

It cannot be too strongly emphasized that, in the case of the older and at present more prevalent plan of the 8-4 school organization, the work in mathematics of the seventh, eighth, and ninth grades should also be organized to include the material here suggested.

The prevailing practice of devoting the seventh and eighth grades almost exclusively to the study of arithmetic is generally recognized as a wasteful

marking of time. It is mainly in these years that American children fall behind their European brothers and sisters. *No essentially new arithmetic principles are taught in these years, and the attempt to apply the previously learned principles to new situations in the more advanced business and economic aspects of arithmetic is doomed to failure on account of the fact that the situations in question are not and cannot be made real and significant to pupils of this age.* We need only refer to what has already been said in this chapter on the subject of problems.

New York State Syllabus in Junior High School Mathematics (1928). The curriculum revision wave which swept the country during the past decade produced a large number of new or revised syllabi, issued by committees, boards of education, examining boards, specialists, associations, and central educational authorities. All of them aimed to be progressive. Not a few were hastily assembled, depending on a dubious array of "paper objectives." Some of them were scholarly and really constructive. Among these documents the one issued by the University of the State of New York is especially complete and systematic. For the study of intuitive geometry it suggests the following topical outline:

I. Index of important terms used in intuitive geometry. [About 120 concepts are enumerated.]

II. Relations.
   1. Important geometric truths. [A list of basic geometric facts is submitted.] At the end of the course, the transition to demonstrative geometry may be attempted in classes of at least normal ability. Whether their proof be assumed or not, the three cases of congruent triangles should be made the basis of a number of the very simplest original exercises.
   2. Important geometric relations and loci.

III. Measurement and applications.
   1. Direct: lines, angles, areas, volumes.
   2. Indirect:
      a. Drawing to scale on squared paper.
      b. Construction of the figure involving the unknown length or distance, using any convenient scale.
      c. Computation of lengths and distances by numerical trigonometry.
   3. Practical measurements: shop, household arts, designs, graphic charts, and diagrams.

IV. Constructions—ruler and compasses required, as well as squared paper whenever helpful; triangles and T-square may be used if available.
   1. Segments of a line of specified length.
   2. Circle of given radius.
   3. Angle equal to a given angle.

4. Construction of triangle, given certain parts: special cases—right, equilateral, isosceles.
5. Parallel lines.
6. Four fundamental constructions: bisection of line, bisection of angle, drawing of a perpendicular to a given line at a given point in the line and from a point outside the line.
7. Rectangle, square, hexagon, octagon.
8. Scale constructions: congruence and similarity.

V. Applications.
1. To shop drawings.
2. Interpretation of blue prints.
3. Making of house plans.
4. Design.
5. Pattern making.
6. Survey plots.

This outline is preceded by a "Basic List of Central Objectives," of which the following have a bearing on intuitive geometry:

1. The fundamental processes and facts of direct and indirect mensuration.
2. Space intuition.
3. The ability to discover and use relationships.
4. The mastery of important mathematical terms, ideas, or concepts.
5. The acquisition of important mathematical skills, habits, and attitudes.
6. The development of important mathematical types of thinking (analysis, generalization, reflective thinking, functional thinking, etc.).
7. Appreciation of the indispensable rôle of mathematics in the modern world, (*a*) as a tool, (*b*) as an organized body of important truths.
8. Appreciation of the ideal of perfection, of absolute correctness and accuracy, of permanent truth, which dominates mathematical teaching.
9. Realization of the long and interesting historical development of mathematics.
10. Appreciation of the beauty of geometric forms found in nature, industry, and all the applied arts.

The syllabus also contains sixteen pages of "Suggestions for Teachers," with reference to the teaching of intuitive geometry, which are concerned with aims and objectives, method, measurement, concepts, facts, relations, and geometric drawing and constructions.

Recent textbook tendencies. Since the textbook is still almost synonymous with the course of study in many American schoolrooms, it is a matter of serious concern to the future of mathematics that our new texts should be constructed in harmony with (1) the best traditions of the subject, (2) the recent recommenda-

tions of competent and experienced specialists in education and in mathematics, (3) the crucial findings of actual classroom experimentation over a considerable period of years.

Any impartial analysis of recent texts along these lines is not very reassuring. Some of the outstanding pedagogic blunders of the past century are still retained with fatal persistence. We seem to learn very slowly from the experience of those who have preceded us. A Pestalozzian organization, "from the simple to the complex," is not uncommon. In fact, organizing principles are usually conspicuous by their absence. No two authors seem to agree on the most worth-while objectives, or on method and sequence. The scene is one of almost hopeless confusion. Perhaps this is unavoidable at the present moment, but it is a situation which is not conducive to increasing the regard for the status of mathematics in our schools.

A group of textbook studies appeared in the *Fifth Yearbook* of the Department of Superintendence (1927), which fully corroborates the preceding statements. An analysis of more than twenty junior high school texts in mathematics revealed that the amount of geometry included ranged "from almost no geometric material to one hundred per cent."[116] Another analysis, of thirteen series of junior high school textbooks, prepared for the same *Yearbook*, led to findings such as the following:

1. The topics selected by authors to meet these objectives [previously stated] vary widely in number and emphasis. *This variation is so great that the conclusion is reached that authors must interpret these objectives differently or else there is no relationship between content and achievement of objectives.*

2. With the possible exception of the first two topics in each of the books there is *no agreement among authors as to the order of presenting the various sections.*

3. Authors seem to *vary less on all these points in the ninth year than in the two preceding years.* This is probably due to the fact that the ninth-grade courses are primarily algebra.

4. Finally, while an attempt has been made to place the facts in as favorable a light as possible, *one cannot escape the conclusion that junior high school mathematics is still in a transition stage* and that *it will probably be some years before definite objectives, content, methods, and results can be agreed upon.*

[116] *Department of Superintendence, Fifth Yearbook*, pp. 202 ff., Department of Superintendence of the National Education Association of the United States, Washington, D. C., 1927.

It is encouraging, however, that an examination of seven junior high school textbook series in mathematics which have been published during the past six years shows a great improvement in certain directions. For example, the geometric topics or "units" are tending to become an integral part of the course and are no longer placed at the end of the book "for optional use." This is certainly a very great victory for intuitive geometry. Again, the illustrative features of the texts are often of a superior order. The "life situations" are becoming less trivial or imaginary. The suggested activities and exercises are a little less fictitious and come somewhat closer to classroom possibilities. The technique of measurement has improved. Not enough provision is made, however, for a *gradual* growth of the geometric abilities. Materials are often "bunched" in a confusing way, showing hasty compilation or a careless arrangement. Finally, all too often, there is lacking the clear-cut evidence of actual classroom trial and experience. Let us hope that the coming decade will bring further progress and will eliminate some of these unnecessary blemishes.

PART FIVE

Classroom Methods and Devices

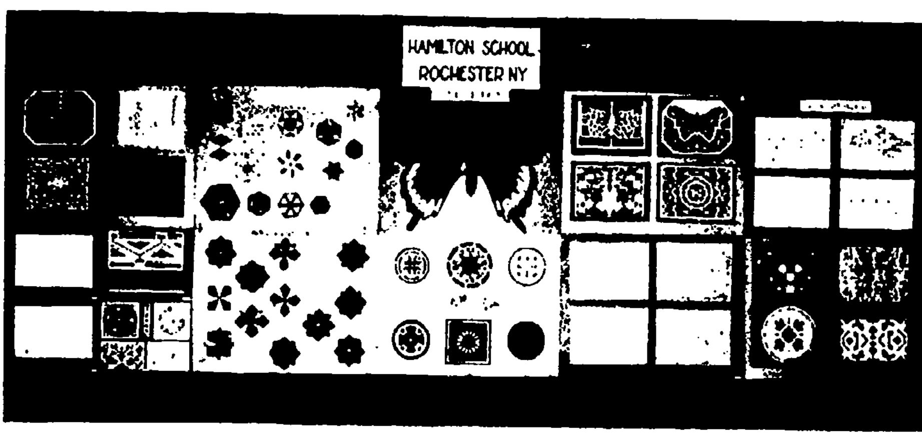
I. The Atmosphere of the Classroom

A. EQUIPMENT OF THE CLASSROOM

The old and the new. Time was when the standard equipment of a mathematical classroom throughout the year was a blackboard, a box of chalk, a set of erasers, and a textbook. To complete the picture, it was only necessary to add the conventional caricature of the weary teacher who was busy all day long "*hearing* recitations," "*assigning* the new lesson," and *correcting* many square yards of faulty exercises "at the board," and hence always covered from head to foot with a layer of chalk dust that threatened to carry her off to an early grave. In this uninviting, truly "dry-as-dust" atmosphere it was indeed hopeless to make mathematics flourish.

Such professional backwardness and indifference is no longer possible or defensible in this age of photography and magazine reading, of movies and high-powered advertising, of rapid travel, of mass education and competing school schedules. Our restless children, harassed on all sides by conflicting interests and by a multi-

tude of daily impressions, are almost forced, in sheer self-defense, to challenge the wisdom of school programs and daily "assignments." "What is it all about?" "What is it good for anyway?" Instead of dismissing such impatient queries as impertinent outbursts of ill-bred adolescents, the alert, modern teacher welcomes them as a means of setting the stage for a period of high adventure and glorious achievement. Under her skillful guidance the classroom gradually takes on the appearance of a new universe. On display boards, on charts and posters, in cabinets and bookcases, on desks and tables, in source books and folders, the necessary illustrative materials and devices are assembled from day to day, until there arises a veritable museum of form, and a genuine laboratory of exploration and thinking.



A School Exhibit Illustrating Symmetry and Design

This illustration is reproduced by the courtesy of Mrs. Mabel Orr, Principal of the Alexander Hamilton High School, Rochester, N. Y., and of Mrs. K. D. Fraher, of the same school.

To experience fully the contrast between the old and the new, in this respect, one has only to enter almost any classroom in an up-to-date junior high school, or to study one of the periodic mathematical exhibits of such a school.

Creating a geometric atmosphere. How the transformation suggested above can be brought about without undue haste or pressure, is described most convincingly by Miss Olive A. Kee, of the Boston Teachers College, in a report published in *The Third Yearbook, The National Council of Teachers of Mathematics.* A few significant paragraphs from this article will serve to suggest the

manner in which Miss Kee approached the task of creating a "mathematical atmosphere."[117] To quote:

Should the mathematics classroom breathe mathematics? Is it possible to develop an arrangement of materials which will make the appearance of the mathematics classroom as distinctive as that of a room where science or history is taught?

It must be admitted that in the majority of cases the visible evidences that a classroom is devoted to mathematics are few and uninteresting. Possibly a set of graphs, which have been cut from newspapers and magazines, and which may or may not mean anything to the students who casually glance at them, is displayed above the blackboard and above eye-level. Or, much better, models of the various geometric forms may be displayed in the cabinet. They may have been constructed by the students, in which case they surely have some meaning. But, with these exceptions, one is likely to find little of significance.

Miss Kee then explains how she undertook to develop the idea of *mathematical posters* to be made by pupils in the schools:

Posters were, of course, rather generally used in the English and hygiene departments of our own and other institutions at that time. No one questioned the fact that a good poster advising us to drink more milk has a more definite appeal than the bare spoken or written message. Posters in the mathematics classroom, however, were an innovation.

The matter was first taken up with those of our students who were preparing to teach in the elementary schools. Considering their interest in the work of the lower grades, the writer saw various possibilities in enlisting them in the poster-making enterprise. Posters of the following sorts were suggested:

1. Those which would show the uses of mathematics in society.
2. Those which carried in their slogan a positive and definite suggestion for the pupil in connection with certain classroom tasks.
3. Those intended to stimulate younger pupils to greater endeavor.
4. Those showing geometric forms in nature and art.

The results were very encouraging, as anyone will testify who, like the present writer, has seen some of the posters made by the pupils during the experiment. We quote again:

From our experiment we are persuaded that stressing the many uses of mathematics may, we do not say will, help us to interest students in it. In any event, there is much evidence to show that students took a livelier interest in mathematics because of the poster project. One of these youngsters in the junior high school said enthusiastically to her teacher, "Why, there's mathematics in almost everything."

[117] Kee, Olive A., "A Mathematical Atmosphere," *The Third Yearbook, The National Council of Teachers of Mathematics*, 1928, pp. 268 ff.

One of our most beautiful posters was that of "Geometry in the Home"—a girl making a lamp shade with hexagonal base. An amusing one was "Know your Proportions"—a luncheon scene with the small son complaining of too much salt. Another had "Math. in the Making, Math. in the Baking" as its slogan, at the risk of having the enemies of mathematics refuse the luscious cake.

Far-fetched? In some cases. Questionable? True. Yet the boy of the junior high school age is ready to challenge and to be challenged. If he tries to meet challenge with accurate information, and if his parents take sides, then the casual public is aroused if not convinced.

In conclusion it may be reiterated that *other features than the work on the blackboard should suggest mathematics.* Is the atmosphere such that all may realize that the teacher loves her subject enough to try to win others to its practice and delights? May even those who dislike the mechanics of mathematics be led to admit freely that the subject lives? *If our classrooms breathe mathematics before and after, as well as during class periods, we may feel that we have taken a step in the right direction.*

B. AROUSING AND MAINTAINING INTEREST IN GEOMETRY

The spirit of the teacher. Even the most complete equipment is useless or harmful when it is not made to serve its real purpose. Of what use is a mountain of musical instruments where there are no musicians to play them? All the paintbrushes in the world, and miles of canvas, are of no avail when the trained hand of the skillful artist is removed from the scene. In the same way, geometry can be taught effectively only by a teacher who understands its background, knows its laws, has mastered its techniques, and is an eager student of its limitless possibilities. Only then does the subject lose its formal, forbidding character, and only then does it convey a vital message to growing boys and girls.

Means of arousing and maintaining interest. Fortunately, the well-nigh universal paucity of pertinent literature and of necessary equipment, so characteristic of former years, is gradually being altered. During the past decade great progress has been made in this direction. The available resources, such as inspirational books and pamphlets, pictures, instruments, study helps, work books, slides, films, source books, club programs, and mathematical recreations, have become much more abundant. Dr. E. Breslich has performed a real service in assembling and making known much of the recent literature bearing on this subject.[118] A great wealth

[118] Breslich, Ernst R., *The Technique of Teaching Secondary-School Mathematics*, Chap. III, University of Chicago Press, Chicago, Ill., 1930.

of valuable suggestions will be found in the pages of *The Mathematics Teacher*, of *School Science and Mathematics*, and of the *Yearbooks* of the National Council of Teachers of Mathematics. The chapters on objectives, model lessons, home-made instruments, mathematics clubs and contests, and on mathematical recreations, in Smith and Reeve's *Teaching of Junior High School Mathematics* [119] are very helpful and supply splendid bibliographies. Dr. William L. Schaaf's *Mathematics for Junior High School Teachers* [120] offers numerous sections on educational values and cultural backgrounds. Woodring and Sanford's *Enriched Teaching of Mathematics* [121] calls attention to supplementary "enrichment" materials. In short, the new spirit which is coming into the teaching of mathematics is in evidence on all sides. And so, with only a moderate amount of enthusiasm and effort, any energetic and open-minded teacher should have little difficulty in creating that most essential of classroom necessities, a mathematical atmosphere.

Typical Classroom Procedures

A. Provision for Motivation

Essential nature of motivation. It is not generally known, as yet, that the newer psychology of learning has established motivation as one of the chief corner stones of education.[122] *Without it there is no real learning.* Hence the naïve attitude of sarcasm, with which conservative or reactionary educators are in the habit of referring to "kindergarten devices," or "soft pedagogy," is not always based on a real understanding of sound psychological principles. *Motivation does not and should not aim to eliminate work.* On the contrary, *genuine motivation is the primary means of encouraging hard work* and of *making it fruitful.* It is vastly different from merely "entertaining" the pupil, as so many seem to think. Hence, real motivation should not be blamed for deplorable evidences of dawdling and inefficiency in some of our school work. *A scientific use of motivation always tends to stimulate a maximum of interest and of classroom efficiency.*

[119] Smith, D. E. and Reeve, W. D., *op. cit.* [90], Chaps. XII, XIII and XIV.

[120] Schaaf, William L., *Mathematics for Junior High School Teachers*, Johnson Publishing Company, Richmond, Va., 1931.

[121] Woodring, M. N. and Sanford, V., *Enriched Teaching of Mathematics in the High School*, Bureau of Publications, Teachers College, Columbia University, New York, 1928.

[122] See Wheeler, R. H. and Perkins, F. T., *op. cit.* [4], pp. 408-420.

Motivating discussions. Each important new topic or unit should be approached by the teacher with a good deal of care. An attitude of expectancy, even of tension, can be created in advance by occasional anticipating glimpses of the new material. When at last the day arrives for a first serious study of the topic, a good deal will depend on the manner in which the "first moves" are made. In the opinion of the writer, which is based on classroom experience, there is no more simple and more effective means of initiating a new topic than that of depending on a "motivating discussion." This is *not a lecture* in which the teacher reveals her superior wisdom. It is, rather, a carefully guided "conversation" with the pupils, in the course of which the new objectives gradually emerge. Thus, if the new unit relates to "measurement," the preliminary discussion may touch on items such as the following: (1) trades, industries, and professions which depend largely on measurement; (2) occasions for measurement in everyday life; (3) the most common measuring instruments; (4) the purposes of measurement; (5) interesting measurement projects suggested by the community; (6) the meaning of accuracy in measurement; (7) the historical development of the common units of length; and (8) what it means to "estimate" lengths.

In the case of very young or very restless pupils, it is well to carry on this background work at the rate of five or ten minutes a day, for several days. When the soil has been prepared in this manner, there is usually very little difficulty in attacking seriously the successive divisions of the topic and in carrying on the activities that are demanded by its objectives.

Insufficiency of the textbook. It is still a cardinal defect of our classroom routine that we expect almost everything from the textbook alone. Naturally, in these days of mass education, the textbook is of more vital importance than ever. It cannot be constructed too carefully. But no single book can provide within its limited dimensions driving power, enthusiasm, background, the stimulus of constant guidance and correction, organic interweaving, and new outlooks; nor can it cause a growth in skills and abilities, the creation of perspective, and a socialized atmosphere—all of which accompany the orderly progress of well-directed classroom activities. Hence, from "book methods" come the permanent limitations of the individualized study technique, so loudly heralded for a while as a panacea, of "directed study," and of mere "recitations."

Intuitive geometry is peculiarly sensitive to the one-sided employment of these methods. It is essentially a *laboratory subject. Its results are not obtained by reading them in ready-made form.* They are derived from an "exploratory technique," still to be described. A large number of witnesses could be quoted in support of this view, both here and abroad. It is significant that the early American books were built on a "lesson" plan, now happily abandoned. In Europe, many textbooks in intuitive geometry never saw a second edition.[123] Some European extremists have even demanded that all textbooks in intuitive geometry be officially *forbidden.* Perhaps one of the chief causes of the slow progress made in the teaching of geometry is to be found in the unwarranted cult of the textbook.

B. THE TEACHING OF A GEOMETRIC "UNIT"

General nature of procedure. At the close of the motivating discussion, the various objectives of the unit are introduced systematically. They usually occupy the attention of the pupils during a period of one or more weeks. The classroom procedure is a combination of what was formerly known as the "development method" and the "laboratory technique." Under this plan, *the whole class is the working unit.* Each pupil has an individual standard notebook or work book, in which the important introductory exercises of each group are entered, while the teacher directs the work orally or from the blackboard. At first, each step is the result of careful questions and of oral responses or conclusions reached by the collective coöperation of the class. As the work progresses, additional problems and exercises provide for the various ability levels. Each pupil is then expected to work at his own best rate and to show definitely that he has a satisfactory comprehension of the day's main objective.

For the sake of variety, or as a rapid check of the mastery obtained, blackboard work may be substituted occasionally or at regular intervals for the individual desk work. If too great a discrepancy exists between the achievements of the best and the "slowest" pupils, it is well to provide supplementary honor work for the accelerated group while retarded individuals are given more time to improve. It is a mistake, however, to expect perfection during the first contact with a new unit. A "100 per cent mastery"

[123] Lietzmann, W., *op. cit.* [23], Vol. II, p. 68.

slogan, if employed *prematurely*, will do much harm. Some skills and abilities are likely to mature slowly, and temporary backwardness on the part of some pupils should not cause undue worry or alarm.

A typical outline. A textbook cannot possibly present its materials in strict conformity to a laboratory plan without becoming too bulky. It must introduce definitions, rules, conclusions, exercises, and summaries at definite points, whether the pupils are all ready for them or not. The textbook can, at best, suggest only the *final product of a lesson series*. The road toward that goal will often differ considerably from that of the book. Hence, the teacher must have in mind a general plan of action that will eventually lead to the desired outcomes. Thus, at the completion of a unit dealing with one of the standard plane figures, such as the angle, the circle, or the triangle, the pupil should finally be able to show his mastery of the following aspects of the work:

Definitions (concepts).	Constructions and related problems.
Notation.	Relationships or facts.
Classification.	Applications.
Measurement.	Supplementary activities.

The order of these subdivisions may often be a different one, but in the main they will provide a fairly complete inventory of all the classroom or home activities that are suggested by a typical geometric unit.

C. THE DEVELOPMENT OF GEOMETRIC CONCEPTS

Underlying psychological principles. At no point in the teaching process is there such widespread haziness as in the field of conceptual training. This appears to be due largely to a lack of information concerning authoritative psychological theories about the formation of ideas, judgments, and concepts, and about the emergence of reasoning processes. Hence we must pause for a moment and review briefly the doctrines of the newer psychology that have to do with these difficult questions. Says Professor Wheeler:

The *direct memory* of a color, a tone, a touch sensation, or any sensory process is an *image*. The *memory of a perception* is called a *concrete idea*. . . . The *memory of similarities* between objects is called an *abstract idea* or *concept*. It is therefore an outgrowth, genetically, of *comparative judging*. *Remembering similarities between objects enables the observer to group*

these objects into classes and leads to processes of reasoning. A scientific law is a concept derived from noting similarities in the behavior of the same object under *different conditions. A statement with regard to any kind of uniformity or generality* is the expression of a *concept that makes possible the rational prediction of future events.* Moreover, *definitions of all sorts are systems of concepts.* The *entire field of mathematics is conceptual.* A person uses concepts whenever he employs any kind of symbol. . . . *The use of concepts and the making of generalizations are the outstanding features of intelligent behavior from the standpoint of self-observation.*[124]

The growth of concepts, as investigated by S. C. Fisher,[125] according to findings summarized by R. H. Wheeler,[126] was observed to be as follows:

The first stage in the development of the concept was the remembrance of *individual figures* largely in terms of concrete *visual, motor* and other *images.* Later on these concrete images became hazy and attenuated; details having to do with the variable features of the objects disappeared while details pertaining to *repeated features remained. Similarities* between the different objects, then, were being *abstracted* from the objects as wholes and separated from the features which made one object different from another. As time went on, these abstracted features were remembered less by means of visual and more by means of *kinesthetic* and *verbal* imagery, until finally the *name* of the figures sufficed as the material of the concept.

Again, Professor Wheeler tells us that:

Intelligent behavior exhibits various degrees of complexity ranging *from perception to reasoning*, each level representing a certain stage in a growth or maturation process, in the life history of the individual. These stages are perceiving, recognizing, comparative judging, and reasoning. Associated with stages one and two is *imagery;* with stage three, *ideas;* and with stage four, *concepts.*[127]

Hence, *sensory processes and images* are seen to be ultimates of the mental life and "may be compared very roughly to the chemist's molecules and electrons."

A more detailed study of perception, judging, the formation of concepts and of reasoning, is given in Wheeler and Perkins' *Principles of Mental Development.* To quote:

[124] Wheeler, R. H., *The Science of Psychology*, p. 146. Thomas Y. Crowell Company, New York, 1929.

[125] Fisher, S. C., "The Process of Generalizing Abstraction, and Its Product, the General Concept." *Psychological Monographs*, 1916, Vol. XXI, No. 90, pp. 5-213.

[126] Wheeler, R. H., *The Science of Psychology*, pp. 147-148, Thomas Y. Crowell Company, New York, 1929.

[127] *Ibid.*, p. 149.

The observational world of the child, at about this stage [three years of age], is a world of discrete objects. . . . The child sees what he can and ignores the rest. . . . When the child matures sufficiently to grasp larger and more complicated situations in their totality, objects then emerge in patterns of knowledge differentiated with respect to their logical interrelationships. . . . When objects emerge from total perceptual patterns so that the apprehension of concrete *relations* between the parts is involved, the process is known as *judging*. The child is able to tell that this object is different from that, smaller, larger, longer, shorter, hotter, colder, higher, lower; or, that one object is doing something to another. . . . Just as perception always involves an inventive aspect *judgments are processes of invention. Each relation is a discovery.* A vast new world opens up to the child when he discovers that objects and situations actually *differ from one another.* Again the discoveries of these differences should always be made in the spirit of adventure and of creative work. . . .

He [the child] is able to recognize *differences* between objects sooner than he is able to recognize *similarities.* The latter process must wait upon further maturation, which appears *from the eighth or ninth year on.* . . . The patterns of experience that now emerge from this field are perceptions of *vast total situations, the parts of which are all interrelated,* as they emerge, in terms of *explicit knowledge of the properties of wholes.* The properties have taken on whole-character in their own right, and can be employed, deliberately, in the processes of thinking. This process is called "generalizing abstraction," *or the formation of concepts.* It involves the explicit recognition of aspects in *common between all the parts of a complex situation.* . . .

Expressed in words the difference between a judgment and a concept is this: A *judgment* is equivalent to saying, "*This* ice is cold," a statement that implies having compared a piece of ice with something less cold. A *concept* is equivalent to saying, "*All* ice is cold," and implies the discovery of a *property common to all pieces of ice. Rules, principles, definitions, axioms, and scientific laws are all examples of concepts.* Judgments, then, are recognitions of *concrete relations between specific objects,* while *concepts are recognitions of abstract relations.*[128]

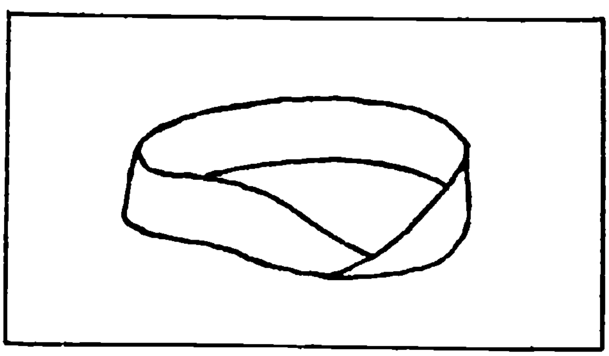
Classroom applications. In the light of this revealing exposition, how futile, not to say grotesque, is the "assignment" of pages of totally meaningless definitions that are to be learned "once and for all," thus dismissing the whole unpleasant job in one grand sweep! But things do not "get learned" that way. Such a procedure is merely a return to the empty verbalism of former generations, so completely discredited by the experience of the ages.

Of course, a child of twelve or thirteen has gone through the preliminary stages of perceiving, recognizing, and judging many

[128] Wheeler, R. H. and Perkins, F. T., *op. cit.* [4], pp. 140 ff.

times, but not, as a rule, in the field of geometry. Hence, it is necessary to make definite provision for these essential steps that lead to the formation of geometric concepts. Perhaps it is now clear how this should be done. Suppose, for example, that the topic of "circles" is under discussion. From a great variety of actual illustrations (e.g. objects, pictures) the *concrete idea* of a circle is obtained. Circles of various sizes may then be drawn on the blackboard or on paper. It is seen that each one may be drawn by a uniform procedure that directs attention to the *similarities* in the structure of circles. In each case we have a *center;* there is a fixed *radius;* the figure is drawn *on a plane surface.* At first, these constituent elements of the concept "circle" remain attached to their visual and motor background. Thus, for a time the pupils should be expected merely to *draw* a circle and to identify its elements in the drawing, or to make a "gesture drawing" in the air. The "verbal imagery" should be added *gradually,* until the pupil can at last give a verbal definition that "suffices as the material of the concept." That is, as someone has said, *a definition should be a summary of experience, not the starting point!*

One more word of caution is necessary. Not every concept can or should be defined verbally. There must always be basic *undefined* terms. Thus, no well-informed teacher will be guilty of perpetuating the time-honored pseudo-definitions of point, line, and plane. Then, too, it is a mistake to demand of beginners an absolutely *final* formulation of all concepts, not to be modified or "unlearned" at a later stage. The history of science and of mathematics is full of contrary warning signals. Many a fundamental concept has been so enlarged in the course of time, because of a wider experience and a growing insight, that the originators of the concept would no longer recognize it. Thus, a Möbius surface [129]



A MÖBIUS SURFACE (one-sided)

[129] A one-sided Möbius surface may be obtained as follows. Take a rectangular strip of paper, turn one of the ends over, through an angle of 180°, and

has but one side, while untutored intuition ascribes two sides to every surface. There are curves that fill every nook and corner of a *surface*.[130] And there are continuous curves that have no tangents. An examination of recent texts on higher geometry shows that the development of concepts is one of continuous growth and hence is a task of extreme delicacy.[131]

In conclusion, it cannot be stated too emphatically that the concepts of a science are its most essential tools. Hence, the successful elaboration of clear geometric concepts will always be one of the most crucial tests of good geometric teaching, and one of its major responsibilities.

D. THE TECHNIQUE OF MEASUREMENT

Providing for the organic growth of skills and abilities. It is now known that "skills," to be effective, must be developed in significant settings. Unrelated skills are quickly forgotten. The following key sentence, to be kept in mind in connection with any "mechanical" technique, might well serve as a motto for many a classroom exercise:

> Practical skill, modes of effective technique, can be intelligently, non-mechanically used only when intelligence has played a part in their acquisition.[132]

Hence, it would seem that *real projects* should always be used as a background for the major techniques of geometry. Owing to the usual limitation of time and available resources, however, this approach is not as ideal as it is thought to be. In most cases the teacher must be satisfied with described, or dramatized, life situations. And there is no reason why the child's *imagination* should not supply, with sufficient vividness, the motivating reasons for otherwise barren, unrelated bits of almost any mechanical routine.

Classroom procedures. In *The Third Yearbook* of the National Council, the writer submitted a detailed study of the tech-

then paste the two ends together. The fact that this surface is really one-sided would quickly be discovered if the attempt were made to use a certain color in painting one of its sides, and a different color in painting the other side. See Lietzmann W., *Aufbau und Grundlage der Mathematik*, p. 27, Leipzig, 1927.

[130] Young, J. W., *Fundamental Concepts of Algebra and Geometry*, pp. 167 ff., The Macmillan Company, New York, 1911.

[131] See Hilbert, D. and Cohn-Vossen, Stefan, *Anschauliche Geometrie*, Berlin, 1932.

[132] Dewey, John, *How We Think*, p. 52, D. C. Heath and Company, Boston, 1910.

nique of direct measurement in the junior high school. It was suggested that the work in direct measurement should be carried on in four distinct, successive steps, as follows:

1. A careful study of the measuring instrument should be made.
2. Adequate practice in "prescribed" measurement must be obtained.
3. Definite provision must be made for the development of accuracy, through "controlled" measurement.
4. The skill thus acquired should then be exercised extensively through suitable applications.[133]

A "prescribed" measurement is one that provides specified numerical data. Thus, we have an illustration of this kind when a pupil is asked to draw a line segment 4½ in. long, or an angle containing 57°. All work in scale drawing with given dimensions, and in the construction of graphs from given data, falls into this category.

In the case of "controlled" measurement, the pupil does not know the final numerical values in advance, but the teacher can check or control the result *by means of the known properties of the figure*. Thus, if the legs of a right triangle are given numerically, the length of the hypotenuse *must* have a certain value. If the pupil measures the successive interior angles of a polygon and then finds their sum, a definite value *must* result in each case. Indirect measurement, when based on scale drawings, involves *both* prescribed and controlled measurement.

The four steps mentioned above can be used in the case of linear measurement, of angular measurement, and of areas and volumes. Thus, when the coördinates of the vertices of a plane figure are given, in terms of specified units, a definite perimeter, a definite angle-sum, and a definite area will be obtained. Similar exercises might be suggested for a three-dimensional domain.

E. THE TECHNIQUE OF DRAWING AND CONSTRUCTIONS

The care and use of the instruments. Much attention must be given at all times to the condition of the geometric instruments. The mechanical details of this important item should be worked out at the very beginning of the course. If the pupils have their own instruments, they must be trained to have them ready for

[133] Betz, W., "The Teaching of Direct Measurement in the Junior High School," *The Third Yearbook, The National Council of Teachers of Mathematics*, 1928, pp. 149-194.

instant use at a moment's notice, without undue noise or confusion. When the school furnishes all the supplies and instruments, a special system of distribution is necessary. The best device that has come to the attention of the writer is that of using special trays, one for each row of seats, with three compartments in each that take care of rulers, compasses, and protractors.

The pupils must be taught how to hold each instrument or how to place it on the paper, and how to perform the required motions. Much time can be saved if drawing boards are available. It is a good plan to have the pupils visit a class in mechanical or architectural drawing, and to bring in samples of work completed in these classes.

The use of the kite and of composite figures. The fundamental constructions are often taught by a purely imitative procedure. Thus, the pupil is shown how to bisect an angle without understanding the reasons for the various motions. The result is always an undesirable mental uncertainty and confusion. Instead, a thorough understanding of these constructions can be developed organically by studying the symmetric properties of the kite, in which case even very slow seventh grade pupils quickly grasp the essentials of the construction problems which it suggests.

Again, the *isolated* presentation of individual constructions is misleading. They should be used in organic combinations. So far, no better plan has been suggested for this purpose than the use of applied design, or of composite figures such as Gothic windows, trefoils, quatrefoils, star polygons, and the like. Not only are all the fundamental construction skills introduced in natural combinations when this method is used, but an immediate check is provided for the degree of accuracy that should be attained at any stage of the work.

F. INVESTIGATION OF GEOMETRIC RELATIONSHIPS

Three modes of approach. Next to the development of clear geometric concepts, the study of geometric relationships in a proper setting is the principal task of geometry. If this work is to be of educational significance, it must be carried on in the spirit of discovery, of exploration, and of adventure. It is not of supreme moment *how many facts* the pupil learns, *but everything depends on how he acquired these facts.*

In this process of exploration, as has been suggested above, there

must be a gradual transition from observation to reasoning. Before we go on, it may be well to recall Carson's distinction between the terms "deduction" and "proof." To quote:

> At this point I wish to suggest that a distinction should be drawn between the terms "deduction" and "proof." There is no doubt that *proof implies access of material conviction, while deduction implies a purely logical process in which premisses and conclusion may be possible or impossible of acceptance. A proof is thus a particular kind of deduction, wherein the premisses are acceptable* (*intuitions, for example*), and *the conclusion is not acceptable until the proof carries conviction, in virtue of the premisses on which it is based. For example, Euclid deduces the already acceptable statement that any two sides of a triangle are together greater than the third side from the premiss* (*inter alia*) *that all right angles are equal to one another; but he proves that triangles on the same base and between the same parallels are equal in area, starting from acceptable premisses concerning congruent figures and converging lines. The distinction has didactic importance, because pupils can appreciate and obtain proofs long before they can understand the value of deductions; and it has scientific importance, because the functions of proof and deduction are entirely different. Proofs are used in the erection of the superstructure of a science, deductions in an analysis of its foundations, undertaken in order to ascertain the number and nature of independent assumptions involved therein.*[134]

Three main roads are open to us in the game of ascertaining or testing geometric truths. Some of these facts will seem certain at once, as soon as attention is directed toward them. This is true, for example, of the basic *positional* facts relating to the intersection of lines or of lines and circles, of symmetric relations, and the like. Such truths are accepted immediately because they possess an *intuitive*, compelling power of conviction.

But there are other truths which, while appearing *plausible* at the outset, are made completely convincing only by a physical or mental "experiment." Many of the metric facts belong to this group, such as the equality of vertical angles, the angle properties of parallels, and the laws of congruence. Here we may resort to a technique of *superposition*,[135] of *motion* (rotation, translation,

[134] Carson, G. St. L., *op. cit.* [63], pp. 25 ff. A more extended quotation bearing on this point is given by Reeve, W. D., in *The Fifth Yearbook, The National Council of Teachers of Mathematics*, 1930, p. 12.

[135] The question of *superposition* and *motion*, in connection with logical geometric proofs, is one of extreme delicacy. Euclid used superposition explicitly but three times. Hilbert eliminates motion by suitable "axioms." Veronese, Bertrand Russell, and others have attacked the use of motion in geometry on the ground that geometry is concerned with empty space which is immovable. Thus, in 1902 Russell wrote that what in geometry is called a motion is merely the

folding), either actually or in imagination. Conclusions thus obtained may also be *verified by measurement*. It should be clearly understood, however, that since measurements are always approximate, real "proofs" can never be obtained in that way.

Finally, there are relationships which must be approached by a process of *analysis* followed by an adequate *generalization*. For example, the rule for the area of a rectangle is not immediately apparent. Hence, we resort to the plan of outlining a rectangle of prescribed dimensions on squared paper. It is then possible to find a definite and uniform relation between the dimensions of *any* rectangle and its area. Let it be remembered, as Dr. T. Percy Nunn has pointed out, that a *single* figure is sufficient in this process of generalization.

In a first course, the process of analysis described above does not

transference of our attention from one figure to another. It appears that Russell's opposition to motion has induced some of our British colleagues to place a wholesale prohibition on this time-honored pedagogic device. In the report on *The Teaching of Geometry in Schools* (London, 1923), there occurs this quotation from Russell's *Principles of Mathematics* (p. 405, published in 1903), relating to the usual congruence of "proofs" by the use of motion:

"'To speak of motion implies that our triangles are not spatial, but material. For a point of space is a position, and can no more change its position than a leopard can change his spots. The motion of a point of space is a phantom directly contrary to the law of identity: it is the supposition that a given point can be now one point and now another. Hence motion, in the ordinary sense, is only possible to matter, not to space. . . . The fact is that motion, as the word is used by geometers, has a meaning entirely different from that which it has in daily life. . . . Motion is a certain class of one-one relations' between classes of points, and 'a motion presupposes the existence, in different parts of space, of figures having the same metrical properties, and cannot be used to define those properties.'"

The report proceeds to give a detailed critique of the related ideas of congruence, motion, and rigid bodies (see pp. 27-35). See also, Heath, Thomas L., *op. cit.* [16], edition of 1908, Vol. I, pp. 226 ff.; Enriques, F., *op. cit.* [21], pp. 98-118; Enriques, F., *Problems of Science*, pp. 100 ff., translated by Royce, Katharine, The Open Court Publishing Company, Chicago, 1914.

The antagonism to motion has recently led to the increasing practice of *assuming all* the congruence proofs. This natural result shows clearly the untenable psychological consequences which accompany a *premature* insistence on "pure" logic in elementary teaching. For it cannot be repeated too often that just as the formation of *concepts* depends ultimately on sensory images, the pupil does not arrive suddenly at ideals of "rigor" by the exacting decrees of the logician. A *transition* is necessary, and during this period the usual assumptions made in connection with superposition are psychologically sound and necessary. It is reassuring that very competent critics in Europe and America are endorsing this psychological attitude. (Thus, an excellent discussion of the transition from "concrete" reasoning to "abstract" reasoning, and from intuition to deduction, is given by E. Rignano in his *Psychology of Reasoning*, Chaps. V and VI; English translation by Holl, Winifred A., Harcourt, Brace and Company, New York, 1923.

ordinarily rest on a *minimum* set of assumptions. But insofar as it employs a *chain of connected thought processes* or *inferences,* it is educationally equivalent to the work ordinarily done in demonstrative geometry.

G. SUMMARIES AND TESTS

Importance of frequent summaries. A comprehensive "unit" will always involve a considerable number of objectives. A young pupil is therefore in danger of "getting lost" unless the road he is following is kept constantly before his eyes. Hence, there should be a growing outline of the work to be placed from day to day on the blackboard and in the notebooks. At regular intervals, careful oral or written reviews should be provided. A great variety of procedures may be followed in "getting a new look at the old materials." Nor is it necessary for the teacher alone to initiate and plan all these reviews. A socialized recitation that causes the pupils to ask each other "any question anyone can think of" is often a more severe ordeal, cheerfully endured, than any teacher would care to impose.

Insufficiency of objective tests. The exaggerated claims of the testing experts, on behalf of the superiority of the "new type" tests, have not always rested on a substantial foundation. A reaction has set in, caused by the evident defects of many of these "push button" testing devices. For example, on many of these tests a mere "memorizer" may score as highly as a real "thinker," thus giving a totally wrong impression of the pupil's true achievement. In particular, these tests glorify *results* rather than *methods*. Hence, they are often an incentive to mechanical drill, instead of being a means of developing or testing *power*.

The progress of the pupil can be appraised adequately only by a painstaking observation of his daily oral and written work, his degree of interest, and his growth in the desirable attitudes and types of thinking. When the limitations of *all* tests are realized, they will be used more intelligently, namely as temporary and not conclusive barometers of insight and growth. Much improvement is still possible in the construction of geometric tests.[146]

[146] Mr. A. B. Miller of the Fairmount Junior High School in Cleveland, Ohio, has for years given carefully constructed and scientifically evaluated objective geometric tests in a large number of classes. This unpublished material, when it becomes available, will probably constitute the most authoritative contribution made thus far.

III. Supplementary Devices and Activities

Manual activities and projects. For the active, "hand-minded" youngster, intuitive geometry provides an impressive number of interesting and valuable activities. Each unit calls for the collection of suitable illustrative material; magazine articles, pictures, newspaper clippings, and graphs must be assembled and put on the display board or must be mounted on charts. Natural objects or models, as well as instruments, must be secured for many of the lessons. Frequent occasions arise for paper cutting, folding, pasting, measuring, and the like. Posters and large graphs, used in connection with the various campaigns of the school, offer a welcome opportunity for those who are "art-minded." Finally, for those who are particularly ambitious, there is always the outlet of large, supplementary projects that call for considerable, long-continued effort.[137]

Source books. For some years, alert teachers have encouraged their pupils to make private collections of interesting mathematical materials that have a bearing on the classroom work. In this way the pupils gradually build up their own "source books" which they often prize more highly than any other feature of their course in mathematics. It is a matter of continued surprise to teachers and pupils how many unsuspected "life values" may be discovered in a supposedly "academic" subject like geometry. Naturally, aside from creating interest, the primary purpose of source books is the impetus which they give to the development of "transfer."

Exhibits. Periodically, the resources of a class, or of several classes, may be pooled for the sake of a common corridor exhibit. The incentive coming from such collective displays is almost beyond belief. Many a pupil in this way gets his first glimpse of the great and permanent values of geometry, and henceforth becomes a real "self-starter" in the subject.

Contests, clubs, and dramatizations. Some adults may be disposed to look askance at certain indirect methods of motivation which a skillful teacher sometimes uses to advantage. Such, however, is the appeal of leadership, of socialized endeavor, of group approval in the lives of adolescents, that the modern school has encouraged the constructive possibilities of these gregarious im-

[137] See *The Third Yearbook, The National Council of Teachers of Mathematics*, 1928, pp. 179-182.

pulses. Mathematical clubs exist in many schools and do valuable supplementary work. Assembly programs have been found an excellent stimulant, as well as a means of correcting false impressions. Playlets, likewise, especially when written by the pupils, tend to add a poetic touch to the prose of the daily routine, and thus point the way to a higher orientation.

Glorifying squared paper. Much of the most profitable work in geometry, including many of its supplementary activities, may now be done with the aid of squared paper. Ever since the days of the Perry Movement and of Klein's espousal of "functional thinking in geometric form," this valuable geometric tool has been enjoying an ever-increasing popularity. Between 1900 and 1912 more than eighty articles or monographs appeared dealing with graphic methods and devices. Of outstanding importance was Professor E. H. Moore's classic paper in *The School Review* (1906).[158] The recent history of graphic representation is too well-known to require restatement at this point. Throughout the school year there is almost constant opportunity to introduce exercises and problems that involve the use of squared paper, as the following list of topics or activities will prove:

1. The construction of bar graphs and line graphs.
2. The direct measurement of line segments.
3. The economical drawing of geometric designs, patterns, and plans.
4. The study of basic geometric ideas (congruence, similarity, symmetry, equality, variation).
5. Scale drawing and the indirect measurement of distances.
6. The study of metric, positional, and functional relationships.
7. The plotting of number-pairs in connection with algebraic graphs.
8. The study of "loci."

If functional thinking can be called the basic melody of mathematics, then squared paper can be called the stage on which this melody is constantly rehearsed and depicted in characters that all may learn to read.

Outdoor exercises. If the Egyptian scribe Ahmes could return for a visit to this earth, and could witness a modern geometry class

[158] Moore, E. H., "Cross Section Paper as a Mathematical Instrument," *The School Review*, Vol. XIV, No. 5, pp. 317-338, May, 1906.

in action, he would be full of amazement at the spectacle. What has become, in the meantime, of this simple science of "earth measurement"? From direct measurement and sense perception it has advanced step by step, until now it is even more efficient in indirect measurement and in purely mental experiments. But let us not forget that this imposing structure rests firmly, and forever, on the reassuring soil of Mother Earth. Let us return from time to time, with our pupils, to this primeval fountainhead of geometric knowledge and skill.

Field trips and outdoor "surveying" exercises tend to reduce the taint of bookishness which often robs the "grammar of nature" of its perennial freshness. However, no attempt should be made to reproduce or imitate the highly technical precision work of the trained civil engineer. A *dramatization* of the necessary steps, with homemade instruments, serves our purposes quite as well, if not better, at this stage. The school premises, or the neighborhood territory, will usually afford ample opportunity for the direct or indirect measurement of heights and distances. In case of inclement weather, or of limited time schedules, the school corridors and the gymnasium will provide temporarily for this improvised field work.

Even a few lessons of this sort will arouse an astonishing amount of interest. And to many a young mind there comes in this way, for the first time, a realization of the tremendous power of a science that has taught man how to reach across continents and oceans, and how to explore the mysterious and beckoning labyrinth of space that holds in its infinite recesses an unending configuration of teeming universes.

CONCLUSION

General summary. 1. We have seen that the geometry of everyday life came into existence through the manifold practical activities that called for a knowledge of shape, size, and position. As these activities were really due to fundamental human needs and were regulated and governed everywhere by the same universal natural forces or laws, the development of geometry was bound to exhibit certain common characteristics in all parts of the world. The first steps toward civilization were accompanied by a growing acquaintance with certain outstanding figures, such as the rectangle, the circle, the triangle, and their corresponding solids. Techniques of mensuration and of construction soon became a necessity. The

ideas of symmetry and equality, of congruence and similarity, which were suggested so constantly by natural objects, were seen to be of basic importance in the manual arts. Perpendicularity and parallelism evolved as inevitable consequences of the gravitational forces.

2. In this natural and practical background geometry remained embedded for thousands of years. Gradually geometric instruments or tools were invented. Units of measurement, at first very crude, became established in all parts of the world. The incessant recurrence of certain geometric processes and constructions, such as were necessary in building, farming, weaving, and other household arts, led to the discovery of the principal metric and positional relationships and facts. And the observation of the rhythms of nature, illustrated so majestically by the cycle of day and night, of the seasons, of the lunar phases, of eclipses, and by the changing positions of the constellations, prepared the way for functional thinking, for the creation of the calendar, for spherical trigonometry, and for a scientific astronomy.

3. Thus it was that when at last the Greeks began to speculate about the *validity* of alleged truths, and hence to develop *deductive* thinking, the materials of geometry were the universal property of mankind. The Greek sages did not have to *create* these materials anew. The basic geometric concepts, however imperfectly formulated, were known to the average adult who plied a practical trade, as were many of the fundamental geometric skills and facts. All this information had been derived throughout the ages from direct contact with the soil and with natural objects or manual activities.

That is, three outstanding facts seem to have emerged from our discussion, which a teacher of geometry cannot afford to overlook if he would do justice to the spirit and the potential value of the subject, at *any* stage of instruction.

First, geometry did not drop from the clouds as a ready-made science. Its evolution was very slow and came about in response to everyday human needs and interests.

Second, the framework of geometry—its basic concepts, skills, and facts—required the prompting and maturing influence of real life situations. For example, generalizations such as we have in the rules of mensuration were the outgrowth, or *final product*, of constantly recurring occasions demanding a definite mensurational procedure.

Third, even after the appearance of Euclid's masterpiece, the

world continued to employ the practical geometric methods of the ages rather than the "scientific" approach created in Greece.

The permanent place of intuitive geometry in the curriculum. In summing up the educational rôle of what is commonly called "intuitive" geometry, we may now submit these two theses:

1. Intuitive geometry is the *geometry of everyday life*. The universal forces that created it are also those which will continue to justify and maintain its place in the curriculum. Geometry developed in response to permanent human needs, and it can be defended only when it reflects the compelling background from which it sprang.

2. If the "scientific" geometry of the high school is to survive as a school subject, it must have a prepared soil in which it can grow. Without the preliminary training, without the body of concepts, skills, and appreciations that a good course in "intuitive" geometry should guarantee, demonstrative geometry becomes, at best, a manipulative game and a "bag of tricks." Very few pupils, if any, can derive a maximum educational benefit from the present arrangement which crowds into a few months during a single school year too many stages of development, each of which is now known to require its own period of maturation.

The rôle of intuition in demonstrative geometry. 1. This discussion would hardly be complete without a brief consideration of the function of intuition in the geometry of the high school. No aspect of geometric instruction is the source of more pronounced conflict and misunderstanding.

Let it be asserted at once that *intuition* plays an indispensable rôle in *demonstrative* geometry. And this statement is true even if we accept a thoroughly rigorous, abstract definition of a "mathematical science," such as the one given by Veblen and Young in their *Projective Geometry*. To quote:

> The starting point of any strictly logical treatment of geometry (and indeed of any branch of mathematics) must then be (1) a set of undefined elements and relations, and (2) a set of unproved propositions involving them; and (3) from these all other propositions (theorems) are to be derived by the methods of formal logic.[122]
>
> Moreover, since we assumed the point of view of formal (i.e. symbolic) logic, the undefined elements are to be regarded as *mere symbols devoid of content, except as implied by the fundamental propositions*. Since it is manifestly absurd to speak of a proposition involving these symbols as self-

[122] The numerals were inserted by the writer.

evident, the unproved propositions referred to above must be regarded as mere assumptions. It is customary to refer to these fundamental propositions as axioms or postulates, but we prefer to retain the term assumption as more expressive of their real logical character.

We understand the term a "mathematical science" to mean any set of propositions arranged according to a sequence of logical deduction. From the point of view developed above such a science is purely abstract. If any concrete system of things may be regarded as satisfying the fundamental assumptions, this system is a concrete application or representation of the abstract science. The practical importance or triviality of such a science depends simply on the importance or triviality of its possible applications.[140]

A classic example of such a development is the one suggested in D. Hilbert's famous *Foundations of Geometry*.[141] Concerning the choice of the "fundamental principles" or "axioms" of geometry, Hilbert says:

The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. *This problem is tantamount to the logical analysis of our intuition of space.*[142]

Now, our "intuition of space" certainly rests on an empiric foundation. Visual, tactual, muscular, and even auditory sensations are the ultimate foundations of the "pure" intuitions, the undefined elements and relations with which we deal in any geometric system. Without them, these intuitions would not arise.[143] This does not mean, of course, that empiric intuition affects the *logical* character of scientifically formulated definitions or axioms.

Again, it is a grave mistake to suppose that even the most abstract logician can formulate a meaningful system of undefined elements and relations, including a group of consistent and mutually

[140] Veblen, O. and Young, J. W., *Projective Geometry*, pp. 1-2, Ginn and Company, Boston, 1910.

[141] Hilbert, D., *The Foundation of Geometry*, translated by Townsend, E. J., The Open Court Publishing Company, Chicago, 1902; sixth German edition, Leipzig, 1923.

[142] *Ibid.*, p. 1.

[143] An extensive literature bearing on this point is now available. See, for example, Mach, E., *Space and Geometry*, The Open Court Publishing Company, Chicago, 1906; Enriques, F., *Fragen der Elementargeometrie*, pp. 1-10, second edition, Leipzig, 1923; Enriques, F., *Problems of Science*, pp. 109-231, The Open Court Publishing Company, Chicago, 1914; Strohal, R., *Die Grundbegriffe der reinen Geometrie in ihrem Verhältnis zur Anschauung*, Leipzig, 1925; Poincaré, H., *Wissenschaft und Hypothese*, pp. 53 ff., German translation, Leipzig, 1914.

independent postulates, without a final appeal to intuition. Sooner or later, he is compelled to come down to earth and to explain his relations by *meaningful* terms.[144] An architect always plans the foundations of a building with reference to the structure which they are to support. In the same way, a system of assumptions or axioms is significant only if it serves as a valid foundation for the superstructure which it is to bear; that is, if the use or *application* to be made of the fundamental assumptions is known in advance. But this anticipatory knowledge of the complete edifice is possible only after a preliminary, *intuitive* survey of the job that lies ahead.[145] For it would certainly be illogical to expect anyone to claim at the outset a *logical* character for this *anticipated* structure which can *be made logically coherent* only by a properly chosen "foundation."

If all this is conceded, it follows at once that geometry is a *science* only insofar as it is concerned with logical deductions from undefined elements, relations, and unproved propositions. The moment we raise the question of the ultimate "truth" of its statements, or the question of its applications, we are at once in the domain of empiric *intuition*.[146]

2. A rigorous geometry in the sense stated above must necessarily become abstract or symbolic. It tends to eliminate the conventional diagrams or figures and to depend solely on a carefully defined symbolism. The reasons for this modern conception of rigor are well known to anyone who has examined the literature on the foundations of mathematics.[147] Moreover, a truly rigorous

[144] See Strohal, R., *op. cit.* [143], pp. 129 ff.

[145] *Ibid.*, p. 130; see also, Poincaré, H., "The Value of Science," *Popular Science Monthly*, September, 1906.

[146] In a lecture by Einstein there occurs the following interesting passage, reminding one of a similar statement made by Bertrand Russell: "In so far as the propositions of mathematics relate to reality, they are not certain; and in so far as they are certain, they do not relate to reality." See Einstein, A., *Geometrie und Erfahrung*, pp. 3-4, Berlin, 1921.

[147] See, for example, Veblen, O. and Young, J. W., *op. cit.* [140], pp. 2 ff.; Young, J. W., *Fundamental Concepts of Algebra and Geometry*, pp. [illegible] ff., The Macmillan Company, New York, 1911; Veblen, O., "The Foundations of Geometry," *Monographs on Topics of Modern Mathematics*, edited by Young, J. W. A., Longmans, Green and Co., New York, 1911; Keyser, C. J., *The Pastures of Wonder*, pp. 44 ff., Columbia University Press, New York, 1929; Keyser, C. J., *Mathematical Philosophy*, pp. 20 ff., E. P. Dutton & Co., Inc., New York, 1922; Pierpont, J., "Mathematical Rigor, Past and Present," *Bulletin of the American Mathematical Society*, 1928, pp. 23-53.

geometry must explicitly state *all* its assumptions and must then "derive all other propositions by the methods of formal logic."

It is virtually self-evident that ordinary high school geometry does not and cannot attain this ideal of logical perfection. For example, any use of a geometric figure is really an appeal to intuition. Entire groups of axioms are usually taken for granted without any specific enumeration. This is true, in particular, of certain axioms of order (also called axioms of position or "graphic" axioms).[18] We also assume tacitly the possibility or the uniqueness of important constructions. Thus, no one expects a high school pupil to *prove* the proposition that a segment has but one bisection point, or that an angle has but one bisector.[19] We assume without proof that the diagonals of a parallelogram intersect, that a segment connecting any point on the base of a triangle with the opposite vertex lies entirely within the triangle, and so on.

The attempts made thus far to transform high school geometry into a rigorous system have failed, and for obvious reasons.[20] The simple truth is that in putting up the *framework* of ordinary demonstrative geometry we cannot get along without the liberal aid of intuition.

3. Finally, it is a common error to suppose that our conventional "proofs" are derived exclusively by "the methods of formal logic." On the contrary, the discovery of a proof by analysis, as well as its restatement in *synthetic* form, is made possible at every step only with the aid of intuition. *Logic can guarantee the correctness of each step, but it does not tell us which step to take next!* That is the function of *intuition*.[21] In other words, each proof involves a *directive* or *organizing* principle which is never a matter of pure logic. The reason why so many pupils fail in *demonstrative*

[18] See Hilbert, D., *op. cit.* [141], pp. 5 ff.; Veblen, O., *op. cit.* [147], pp. 5 ff.; Fladt, K., *Elementargeometrie*, Part II, pp. 3 ff., Leipzig, 1928.

[19] For a simple, rigorous proof of these propositions, see Thieme, H., *Die Elemente der Geometrie*, pp. 27-28, Leipzig, 1909.

[20] This is shown, for example, by the unpedagogic character of the splendid texts published in Italy, by the total oblivion that has overtaken B. Halsted's interesting *Rational Geometry* (John Wiley & Sons, Inc., New York, 1904), and by the forbidding appearance of such excellent texts or studies as those of Thieme, of Killing and Hovestadt, of Weber and Wellstein, and, to a certain extent, of Fladt.

[21] It was H. Poincaré, especially, who stressed this point again and again (see references given above). Other French writers have strongly endorsed this view, notably Jules Tannery. F. Klein's constant emphasis on intuition is too well known to require explicit comment.

geometry is to be found precisely at this crucial point. Instead of developing a primary, organic understanding of the whole situation, they get lost by concentrating attention on isolated steps. It is the old error of the mechanistic association psychology, which has been exposed so relentlessly by the newer point of view. What we need is an *organismic logic* that replaces the old piecemeal arrangement of "steps" by a configurational outline or plan, in the light of which the steps can be understood.[152] Each demonstration presupposes an exploratory process, an attitude of discovery. And hence, a geometric proof, as was pointed out long ago by Kroman, Hölder, and others, always represents a "mental experiment."[153] Such a view, however, as can readily be seen, amounts again to a constant dependence on *intuition*.

Thus it is that demonstrative geometry, to use a familiar phrase due to Professor Max Simon, is really a "mixture of intuition and logic." It would, therefore, be a wise policy on the part of high school teachers to make a thorough study of the ways in which intuitive and demonstrative geometry are interrelated, an dto adopt a new attitude of appreciative support with reference to the potential value of junior high school mathematics.

A final appraisal. Intuitive geometry, as we have seen, offers a unique field of training in that it exhibits the characteristics of both the natural and the purely mental sciences. By slow degrees it transforms the crude "spatial" sensations or images of childhood into the scientific foundation needed in later life. It leads from immediate contact with "things" to an inner world of thought, of precision and beauty. It builds for the mind an ever-larger home by extending its horizons from the narrow confines of the classroom and the neighborhood to ever-receding nebulæ that float as uncharted islands in outermost space. And it achieves these astounding results with the aid of man's most precious gift, namely, his

[152] R. H. Wheeler has made a real contribution by his useful and lucid discussion of "atomistic logic" as compared with the "logic of unity," or "organismic logic." See Wheeler, R. H., *The Laws of Human Nature*, pp. 40 ff., D. Appleton and Company, New York, 1932.

[153] K. Kroman's ideas about "mental experiments" in geometry, dating back to 1883, are discussed by O. Holder in his *Anschauung und Denken in der Geometrie* (p. 10, Leipzig, 1900). In his exacting recent treatise, *Die mathematische Methode* (Berlin, 1924), Holder again analyzes the nature of these mental experiments in an interesting manner (pp. 27 ff.). A most helpful and readable presentation of the Kroman theory, and of related questions, is found in Rignano, E., *The Psychology of Reasoning*, pp. 121 ff., *op. cit.* [155].

ability to *think*. Geometry is indeed the world's most potent laboratory of thinking, and thinking of a type that can be fruitfully applied in myriad directions. As such, it will continue to make its appeal to those who would be truly educated. For back of our industries, back of our inventions and our social institutions, there stands—as the originating and organizing force that keeps all things moving—the trained mind of the *thinker*. "Our sole dignity consists in *thinking*."

COHERENCE AND DIVERSITY IN SECONDARY MATHEMATICS

A PERSONAL PHILOSOPHY BASED ON THE OPINIONS OF MANY LEADERS IN EDUCATION

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We cannot know, but we can think. We can never fully know what is the true end of man: we can never know the precise purpose of the secondary school; we can never know the part which mathematics ought to play in training the youth of this nation. These matters rest ultimately upon philosophy, and there are many philosophies. The answers vary according to the philosophy we choose. If we choose that one which yields the most satisfactory answers, we must realize that this very satisfaction, however obtained—from standardized tests, subjective opinion, the experience of the ages, common sense—derives its ultimate validity for us from *some* philosophy. Our choice of answer, of what to us is satisfying, reflects indeed our choice of *a* philosophy. Our neighbor may choose differently. As between the respective merits of these philosophies, who can arbitrate?

Even if we can never surely know, a common philosophical notion prompts us to the belief that if we would live, be good citizens and good teachers, we must try *to discover by thinking* the most satisfying philosophy concerning the education of boys and girls in secondary schools, referring particularly, but by no means exclusively, to the part to be played by mathematics. No philosophy will satisfy us which will not also satisfy the boys and girls as we think of them. This chapter aims to present a philosophy which recognizes ancient good and new endeavor, and attempts to reconcile them. Confident that anyone else will find it as difficult to prove the present philosophy wrong as for the writer to prove it right, it is submitted in the hope that it may provoke further thought.

An eclectic philosophy. The steady accumulation of statistics, results of experiments, and philosophical inquiry all indicate that the wheels of progress are turning. The writer has gathered the opinions of leading teachers of mathematics in secondary schools and colleges and of educational philosophers over a period of years as a necessary part of his professional work, and has reënforced these opinions recently by statements written in answer to his questions. The queries were in the form of a questionnaire which was a bit unusual in that it allowed the answerer great latitude in discussing any idea suggested by the questions, and even encouraged such diversity of response. This departure from accepted practice was prompted by the fact that the number of persons questioned was too small to warrant reliable statistical treatment. When large numbers are involved, the errors introduced by the necessity of telescoping one's opinion on a complicated question into a convenient catchword—*hot, tepid,* or *cool*—tend to cancel one another. In this case, the small numbers involved and the broad philosophical import of many of the questions caused the writer to give preference to a procedure which would encourage the answerers to discuss their opinions in greater detail. These opinions, somewhat conflicting, yet often surprising in their unanimity, have contributed largely to the philosophy of mathematics in the secondary school embodied in this chapter. The responsibility for the final statement belongs to the writer alone, but credit for the majority of the ideas is due to numerous authors whose writings have become so much a part of his thinking that he dares call nothing his own for fear of unconscious plagiarism; and due also to a group of persons who have contributed so recently to his thinking that the details of their contributions can be distinguished and identified.[1]

[1] The list of contributing persons is given here with grateful acknowledgment: C. R. Adams, Brown University; Gertrude Allen, formerly Oakland (Cal.) High School; F. L. Bacon, Evanston (Ill.) Township High School; Bancroft Beatley, Harvard Graduate School of Education; W. H. Betz, Alexander Hamilton High School, Rochester, N. Y.; E. R. Bowker, Public Latin School, Boston; E. R. Breslich, University of Chicago; T. H. Briggs, Teachers College, Columbia University; P. T. Campbell, Superintendent of Schools, Boston; W. W. Charters, Ohio State University; P. W. L. Cox, New York University; Maurice Crosby, Smith College; A. A. Douglass, Pomona College; H. R. Douglass, University of Minnesota; W. F. Downey, English High School, Boston; L. P. Eisenhart, Princeton University; G. W. Evans, Lynn, Mass.; J. P. Everett, State Teachers College, Kalamazoo, Mich.; E. N. Ferriss, Cornell University; W. B. Fite, Columbia University; H. D. Gaylord, Browne and Nichols School, Cambridge, Mass.; A. L. Gould, Assistant Superintendent of Schools in charge of Intermediate Schools,

The old and new contrasted. We know that the secondary school arose in this country as the first stage in the training of preachers and teachers. From a prescribed curriculum for a select few, it has developed into a diversity of curricula and elective courses for all. Prospective preachers and teachers are scattered among our pupils still, but now in an almost unobserved minority. Failure to pass indicated originally that the pupil was not adapted to the course and the pupil was dropped; failure is now interpreted as indicating that the course is not adapted to the pupil, and the course is dropped, or modified. The old courses have lost some useless matter and received slight amounts of new material, mostly good. A general relaxation in the degree of difficulty of all courses favors the average student of to-day, less gifted mentally than the average student of fifty years ago. The provision for students of even lower mentality, in mathematics at least, is almost wholly inadequate and grossly unfair to them. Equally unfair, and much more harmful to society generally, is the inadequate provision for students of superior ability. Can we not recognize and preserve the good in both the old and the new philosophies, and devise means to serve better both the old and the new types of students, neither of whom at present receives his due?

Do we not owe it to the superior student to provide him with courses of study which are substantial, continuous, and coherent; courses which are sufficiently challenging to develop the best that is in him? Must acceptance of the philosophy of a period of try-out in the junior high school years mean little definite progress in those years? What try-out is there in dawdling; what valuable prognosis for future guidance?

Was the elective system ever intended to sponsor the kaleido-

Boston; W. C. Graustein, Harvard University; Henry Harap, Western Reserve University; M. L. Hartung, University of Wisconsin; E. R. Hedrick, University of California at Los Angeles; Martha Hildebrandt, Proviso Township High School, Maywood, Ill.; L. C. Karpinski, University of Michigan; O. D. Kellogg, Harvard University; R. E. Langer, University of Wisconsin; W. R. Longley, Yale University; E. B. Lytle, University of Illinois; C. H. Mergendahl, Newton (Mass.) High School; A. B. Miller, Fairmount Junior High School, Cleveland, Ohio; W. M. Proctor, Leland Stanford University; W. D. Reeve, Teachers College, Columbia University; W. H. Roever, Washington University; Vera Sanford, Western Reserve University; J. Shibli, Pennsylvania State College; Clara E. Smith, Wellesley College; R. R. Smith, Central High School, Springfield, Mass.; F. T. Spaulding, Harvard Graduate School of Education; J. A. Swenson, Wadleigh High School, New York City; E. H. Taylor, Eastern Illinois State Teachers College; Mary E. Wells, Vassar College.

scopic programs of study so commonly permitted to-day? Ought freedom to choose the field in which a pupil will concentrate his efforts ever mean freedom to choose to concentrate on nothing at all? And how unfair to the pupils, especially those who look forward to professional life, who will later discover that overproduction in recent years has included overproduction of candidates, underprepared for the professions! The candidates themselves are becoming aware of this, in the colleges at least, and from necessity are embracing the virtue of real concentration in a chosen field. President Lowell of Harvard and others who have long advocated that students in college address themselves more strenuously and whole-heartedly to one subject, with an eye to substantial mastery of that subject, have been abetted in the latter stages of their campaign by the pressure of necessity upon the students themselves to follow that very course. There is a lesson here for the secondary schools. It is not intended, of course, that this concentration of effort upon a chosen field shall imply a total neglect of other fields; it demands reasonable distribution as well as concentration.

We have as yet devised very little material appropriate to the needs of those students of inferior mental capacity for whom we desire to provide some proper schooling on the secondary level. We prefer that their entrance into industry or commerce be delayed so that they may enjoy the environment of the school as long as possible. We believe also that abstractions and generalizations are possible at all levels of mentality, though in varying degrees. Since in the case of some students algebra is a convenient vehicle for the conveyance of generalizations and abstractions, it was natural that we try to employ it for inferior students[2] as well as for superior ones. But experience has led us to modify our practice, so that to-day our offering to inferior students is but faintly reminiscent of algebra and ignores generalizations and abstractions almost entirely. By all means teach them the elements of formula, graph, and equation; call these "algebra" if so desired, but recognize that our treatment of these topics for inferior students includes little of the true essence of algebra. Let us still encourage them to generalize, to think abstractly; if not algebraically, then in some other medium. If our recent efforts convince us that the proper vehicle for generalizations and abstractions by these pupils is almost certainly not algebra, let

[2] The phrase "inferior students" is intended to denote mental inferiority only, with no implication concerning the students' other qualities.

us then devise other material better suited to this purpose. This material might be derived from simple geometric relations, from numerical relations in arithmetic—generalizing without employing the general numbers of algebra—or from non-numerical material essentially social or linguistic. The artificial languages familiar to us from certain mental tests are suggestive in this connection. It may be objected that mathematics cannot fairly be asked to develop non-mathematical means of teaching generalization and abstraction. Is this, however, less appropriate to mathematics than that part of our instruction in the eighth grade wherein we impart the details of certain social customs connected with the drawing of checks, promissory notes, taxes, and insurance without making any significant contribution to arithmetic itself?

In place of algebra for slower minds, we often prescribe commercial arithmetic, bookkeeping, or business arithmetic. Are these subjects easier than elementary algebra and are these the people who will later keep the accounts of commerce and industry? Is there not greater justification for teaching the arithmetic and simple algebra of business to those who will finish the senior high school or even go to college before entering business?

Differentiation in curricula. If the arithmetic of milk and soil analyses is apparently useless to city boys, has it no justification from the point of view of appreciation? If there is any mathematics in home economics, is it properly reserved for girls only? Do not the mathematics of the home and of business concern us all? Is the geometry of bridges, silos, belts and pulleys likely to have less ultimate value for students preparing for college than for students in industrial courses? For which group is an appreciation of the geometry of Gothic window tracery more important? Both classes of students are preparing for life, and both are actually living. A topic of immediate utility to one has the possibility of great cultural value to the other. Either outcome has been commonly held to justify a topic in mathematics. What justification is there then for marked diversity in the mathematical content of different curricula? It is true that utilitarian values and appreciatory values motivate students in differing degrees, but other aspects of the topic, or of the total learning situation, usually transcend the utilitarian and appreciatory values in their motivating appeal.

Evidently the present differentiation in the mathematical con-

tent of our various curricula is of no great consequence to our pupils. For no one seriously regards the student's choice of curriculum in secondary school as indicating his final vocational destiny. It is a commonplace that bright students in any curriculum can easily go to college. From the *Biennial Survey of Education for 1922-24*,[3] we learn that 14 per cent of high school students elect the commercial curriculum, 6 per cent (boys) elect industry or agriculture, and 5 per cent (girls) elect home economics. The other 75 per cent elect an academic course. This means that they are preparing for college, for scientific school, for normal school or teachers college, or are "general students." Does anyone believe that these figures represent, even approximately, the final vocational apportionment of these boys and girls? Granting that the distribution of pupils in curricula ought to accord more nearly with their final vocational destination, and that there ought to be significant differences between curricula—presumably in the various courses which distinguish one curriculum from another—there is almost no justification for providing different courses in the subject of mathematics for different curricula. A differentiation according to mental capacity, however, has much to commend it. We tend now to ridicule the stilted phrases of "business English"; in business, as elsewhere, we demand simply English. For mathematics, similarly, regardless of vocational objective, we need simply mathematics.

The generalizations and abstractions of algebra and demonstrative geometry are for anyone who likes his abstractions and generalizations in that form, wholly apart from any question of transfer, though that too is germane if not pressed too far. For the purposes of everyday living, the bulk of our algebra and demonstrative geometry is relatively useless; in fact, hardly 5 per cent of all students enrolled in the ninth grade will ever make serious vocational use of either subject. For only 25 per cent of all these pupils will go to college, and of these less than 20 per cent will concentrate on mathematics and the mathematical sciences. The assumption is here made that all these concentrators make later use of mathematics, which is most unlikely. Since the purely utilitarian value of algebra and geometry is so low, diversification of courses in mathematics according to varying vocational objectives

[3] Chap. XXIV, "Statistics of Public High Schools, 1922-24," Table 23. More recent data are not available. This is Bulletin 1926, No. 23, published by the Bureau of Education, Department of the Interior, Washington, D. C.

seems doubly futile. Let us provide the same course in demonstrative geometry and algebra for all bright students who wish to study it, whatever their chosen curriculum. To inferior students much of this algebra and geometry is denied, though many of the educational outcomes of these subjects can be preserved for them in other forms, and ought to be.

The relative uselessness of most secondary mathematics need not be construed as an indictment of teachers of mathematics or of the subject itself. This very uselessness is part of the essence of mathematics and is matched in only slightly less degree by the other subjects of the secondary school. A proper defense of mathematics rests for the most part upon certain cultural values and upon indirect values of a disciplinary sort which can be substantiated only with the greatest difficulty, though this difficulty in itself must not be regarded as any impairment of the values in question.

The proposal to differentiate courses in mathematics according to mental ability rather than by curricula may seem to deny the equality demanded by our democracy. Democracy and equality have been synonyms so long that it is difficult now to distinguish them. True, we no longer insist that all men are created equal. Having come to recognize that individuals differ in some respects, we are led to a new interpretation of democracy in terms of equal opportunity for the development of each individual. But the old flavor lasts. Our educational system bulks large in politics, and ought to; but the full import of this new democracy still escapes our political demagogues. We do not actually provide this equal opportunity of the new democracy to either the slow or the quick.

The suggestion is made above that try-out courses in the junior high school ought to be more substantial, not only to furnish a genuine try-out of the pupils in these courses, but to insure efficient use of this time by all pupils. For superior students we ought to devise a more coherent, continuous, and substantial course of study in mathematics embracing all grades of the secondary school; and for inferior students another course of study, often of quite a different sort, but forming also a coherent, continuous, and substantial program which these students can follow with profit. Each of these courses of study would contemplate the possibility that students might not complete them. They would be devised as far as possible to give in each succeeding grade the most important subject matter in mathematics for the pupils in question. This plan does not deny

the value of separate curricula for students looking forward to commerce, industry, or other specific vocation. It asserts merely that the vocational differentiation between curricula ought not to apply in any large measure to the mathematics of these curricula; that in mathematics diversity of vocational aim is much less significant than differences in mental power.

These matters thus briefly considered, though supported by a weight of expert opinion, are admittedly contentious, the contention lying between rival philosophies. Let us see more in detail just what they imply for pupils in secondary schools, whether in academic, commercial, or industrial courses. Let us begin with the junior high school.

Junior high school mathematics. An article in *The Second Yearbook, The National Council of Teachers of Mathematics*, published in 1927,[1] describes the results of experiment and the experience of many teachers to show that with only one year of arithmetic divided between the seventh and eighth grades, and with the remaining time devoted to substantial work in geometry and algebra, the pupils not only lost none of their meager proficiency in the fundamental operations, but showed a marked gain in the ability to solve problems. This gain resulted undoubtedly from the algebra they studied in the first half of the eighth grade, a full half year of it, given purposely in the first half year that it might affect favorably the arithmetic of the second half year. A more recent study[2] of the relative merits of old and new type courses in the junior high school years corroborates this earlier finding; not only have the admittedly meager results of our instruction in arithmetic not been damaged by the intrusion of geometry and algebra, but, presumably also, these newer subjects have contributed something of value on their own account. For superior and average students we can surely[3] give a half year of informal geometry in the seventh grade, and probably even for students of inferior ability. Almost all students welcome the appeal of geometry to manual dexterity and to concrete representation, and for students of inferior ability

[1] See the article by the author, "Effects of Varying the Amounts of Arithmetic, Geometry, and Algebra Studied in the Junior High School."

[2] Beatley, Bancroft, *Achievement in the Junior High School*, Harvard Studies in Education, Vol 18, Harvard University Press, Cambridge, Mass., 1932.

[3] For substantiation of this point, consult the article mentioned above by the author in *The Second Yearbook, The National Council of Teachers of Mathematics*, 1927.

it is particularly appropriate, affording relief from the constant reference to abstractions in arithmetic. Unfortunately the geometry most commonly taught in the seventh grade to-day is largely confined to the mensuration formerly included in arithmetic. Besides the mensuration, there is much that can be done with compasses, protractor, scissors, and paste pot to teach the important facts of geometry, including solid geometry. The construction of cardboard models of prisms, cylinders, pyramids, cones, and the three simplest of the five regular polyhedrons is well within the range of ability of pupils in the seventh grade. Deciding where to put all the flaps for pasting demands spatial imagination, but no more than the pupils can summon. A whit more patience and a somewhat longer span of attention make possible the construction of the dodecahedron and the icosahedron also. The geometry of loci is well within the range of seventh grade pupils; we need only suppress the technical term "locus" to disclose a vast field of simple and significant geometric endeavor. Slow students who would get little from more arithmetic derive a great deal of profit from informal geometry, and what they get is suited to the brighter students also.

That we can well afford to give more attention to geometry in the junior high school is borne out by the answers of many educational leaders (see footnote 1) to the question, "Could we double the usual dose of geometry in Grades 7 and 8 (including three-dimensional figures) without lessening the return from the arithmetic?" Following are some typical answers:

Probably yes.

No. Increase the geometry by about fifty per cent, but not double. Such a change might be very desirable.

Yes. Very readily.

We could probably increase the amount of geometry taught in Grades 7 and 8 including the solids, without lessening the return from arithmetic. I doubt if we could double the amount.

In our intermediate schools the work in 7B and 8B is almost entirely intuitional geometry.

Yes. It would be difficult to reduce what is already so near to zero.

No. We do not have a sufficient amount of arithmetic in increasingly significant and complex situations for Grades 7 and 8. There is an abundance of useful arithmetic which is still untouched.

No. The returns from arithmetic are poor enough; those from such algebra and geometry as are taught in our grade systems seem even poorer.

Why double? We have been giving a course in intuitive geometry extending through Grades 7A, 7B, and 8B during the past twelve years *without reducing the orthodox arithmetic content.*

Yes. In the junior high school, geometry of three dimensions should be kept before the pupil even when two-dimensional figures are studied. Arithmetic will be helped by geometric problem material because it is real to the pupil. The "usual dose" of geometry in Grades 7 and 8, it seems to me, is very small.

About one-third of the replies denied without further comment the possibility of *doubling* the present amount of geometry. Most of the other replies indicated the desirability of increasing the present offering in geometry.

It seems to be generally agreed that the geometry in Grades 7 and 8 ought to be mainly factual, and outside geometric facts to include nothing of the methods peculiar to demonstrative geometry. In response to the question, "What important geometric facts are better withheld from junior high school students and postponed until the tenth grade or later, and why?" the following reply was received, which summarizes most of the others also:

It seems to me that hardly any geometric *facts* need be withheld. Any ninth grader can deal successfully with the facts of congruence, similarity, angle sums for polygons, angles of a stripe, and so forth. What needs to be withheld is the method of geometry, its bases, its logical development. He is not mature enough to profit by or to comprehend a discussion of these matters until the tenth, eleventh, or twelfth grade.

Some of those questioned made special exception of incommensurables and one or two other of the more difficult geometric facts, but on the whole they agreed that so long as the geometry of the junior high school confined itself to factual material, there was little to limit it. One person, while disclaiming any interest in a unit of demonstrative geometry in the junior high school, would endeavor to show pupils the nature of a "proof of reasoning," though presumably not in the seventh grade. We shall return to this idea.

The geometry and arithmetic of the seventh grade ought to contain applications to commerce, industry, agriculture, the arts, and the home regardless of the future vocational specialization of the individual pupils. Little need exists in this grade to provide markedly different subject matter for bright and dull pupils; the difference in emphasis suggested by separate sections for bright and dull ought to suffice.

Grade 8 mathematics. In Grade 8 the case is different. The duller pupils here will get little from algebra but formula, equation, graph; and of these they will master little beyond the technique.

A formidable difficulty to be overcome in providing diversified content in algebra for bright and dull pupils is that the relatively useless "tool" aspect of algebra—the various techniques—is comparatively easy to teach, whereas the appreciation of algebra as a generalization of arithmetic, and its symbolism as a most expressive language, is very difficult to teach and not easy for pupils to grasp. That part of algebra which is of greatest importance to the "general reader" is much harder to acquire than the technique of algebra, which hardly one in twenty pupils will ever need to use. It would seem practically futile to try to give the slower pupils this appreciation of the real meaning of algebra. Either we must excuse them from further instruction in algebra beyond formula, equation, graph, or else we must devise wholly new material by which we may give them an appreciation of generalization and of a symbolic language. Here in the eighth grade we come for the first time to diversification in subject matter according to mental ability.

A genuine prognosis. The eighth grade has long been a weak spot in our schools, notorious for dawdling, whether under an 8–4 or a 6–6 plan. Under the 8–4 plan, dull students were encouraged to remain and receive their diplomas; consequently the mathematics of the eighth grade contained little that was vital and new—merely additional social applications of arithmetic. If some algebra was included, everyone knew that it was to be repeated in the following year. This persists to a lesser degree under the 6–6 plan. It is at this point that the teacher's earlier doubts concerning a pupil's aptitude for advanced mathematical and scientific work are likely to be reinforced. Here then is a crucial point in the pupil's career. A substantial half year of algebra in the eighth grade, not merely a dabbling with formula, graph, and equation, but a more thorough treatment of these subjects, together with a large amount of work with problems and in the four fundamental operations with negative numbers, would give us at this point a reasonably good indication of the future mathematical promise of the individual student. In short, a program is proposed that approximates in degree of difficulty the traditional first half year of algebra, though following perhaps a different order of topics and a different method of treating them.

Some such prognosis of the pupil's likelihood of success in mathematical and scientific work is due the pupil, the school, and society. If we are to offer a course claiming to have reasonably

reliable prognostic value of this sort, then this course must do what it pretends to do, sift out those who can from those who cannot. The time so spent ought to yield a reasonably sure result and ought in no way to be considered time wasted for those who pass the test and are allowed to continue. The meager course in algebra so commonly a part of eighth grade instruction to-day is neither one thing nor the other. It is not difficult and substantial enough to be an adequate selective sieve, and for those who pass through its meshes it yields so slight a return for time spent that the matters studied are repeated in a later grade. If a subject is not substantial enough to be considered a suitable foundation for later study, how can it serve as a prognostic guide to success in that later study? If we are really sincere in demanding a try-out in Grade 8, then the try-out must be genuine.

By the end of the seventh grade we shall undoubtedly recognize many cases in which it will be unnecessary and unfair to subject the pupil to the genuine try-out in algebra just mentioned. And others who embark upon this substantial course will shortly see the wisdom of quitting before it is over. However, no one wishing to persevere to the end should be denied. The advantage of a substantial try-out in algebra in the first half of the eighth grade would be positive as well as negative. Those who passed could be labeled with considerable assurance as likely to succeed in later mathematical and scientific work. In our present course for the eighth grade we have no such assurance, and find ourselves hampered in later courses by students who cannot do the work.

This choice of algebra as a criterion for later success in mathematics is supported (though by no means unanimously) by the opinions of college teachers of education and by teachers of mathematics. In reply to the question, "Which aspect of junior high school mathematics, intuitive geometry or elementary algebra, is more valuable in discovering a pupil's aptitude for advanced mathematical and scientific work?" three-fifths answered "algebra." One-fifth said "geometry, because of the 'original thinking' it demands," or made equivalent response. Even if they believe that algebra properly requires more abstract thinking than intuitive geometry, they undoubtedly expect little of it in the algebra of Grade 8, for they expressly reject this algebra as "merely dabbling with symbols." The writer agrees that this description of the algebra given is unhappily accurate in too many instances; unfortunately, how-

ever, the dabbling spirit here alluded to is present also in the geometry. From this point of view, neither serves as a reliable measure of the pupil's aptitude. Perhaps the last fifth of the replies had this in mind when expressing doubt about which subject was to be preferred. Much less doubt was expressed by those who actually teach mathematics; almost all these teachers preferred algebra in the eighth grade as a measure of future mathematical power. Of course the thing to do is to make objective trial of the matter. Inasmuch as algebra has traditionally, or till very recently, demanded a degree of sustained thinking not yet commonly expected in intuitional geometry, and inasmuch as most colleges and technical schools—Princeton and the Massachusetts Institute of Technology, for example—would testify that students in freshman calculus and analytic geometry are generally hampered by a lack of mastery in algebra, and practically never by deficiencies in geometry, it seems not unreasonable to look to a substantial course in algebra in the eighth grade as a significant index of later mathematical proficiency.

This insistence on algebra ought not to obscure the need of further work in arithmetic in Grade 8 for pupils of all degrees of ability. The solving of problems demanded by this arithmetic is much helped by the algebra which has preceded, more so by the substantial course than by the skeleton course for inferior students. There seems little reason to prescribe more arithmetic for this latter group of students than for the average and superior students, but they might well be given more geometry than the others have at this time, including an amplification of similar triangles and proportion to embrace the most elementary use of trigonometric tables. Educational leaders, recipients of a questionnaire, replied almost unanimously that the arithmetic of adult life usually taught in the eighth grade is harder than the geometry of compasses, scissors, and paste pot commonly recommended for the junior high school.

Differentiation according to ability. The diversification in subject matter of mathematics recommended for Grade 8 is briefly this: a substantial half year of algebra and a half year of arithmetic for students of average and superior ability; for inferior students, as much of formula, graph, and equation as they can accept, replacing with geometric material the difficult parts of the algebra appropriate to the other students, and following this with a half year of arithmetic. The arithmetic will be the same for all students

except for slight variations between sections of different ability. The extra geometry for the inferior students anticipates in slight measure the more intensive course in geometry which the other students will take later. Some of the generalizations and other values of the algebra withheld from the inferior students will be served up to them in non-mathematical form in Grade 9, together with a non-mathematical treatment of logical argumentation and other values supposed to inhere in demonstrative geometry. By the end of Grade 8 every student, whatever his ability, will have had the opportunity to master all the mathematics which, in the restricted meaning of the term, is considered useful in adult life to-day.

Grade 9 mathematics. The mathematics of Grade 9 and later grades must be justified mainly by reference to cultural and disciplinary values, both difficult to substantiate, for we have already shown that not over 5 per cent of all pupils enrolled in Grade 9 will make direct use of algebra above the elements of formula, graph, and the solution of simple problems by means of one equation in one unknown. True, we cannot know just which pupils in Grade 9 constitute that 5 per cent, but we can be reasonably sure that none are to be found among the students of inferior ability.

From various sources it seems clear that pupils with an I.Q. less than 110 will derive little benefit from a study of algebra which goes much beyond formula, graph, and simple equation. Separate objective studies by M. V. Cobb, I. J. Bright, O. A. Wood, and I. N. Madsen [7] give an I.Q. of 105 as the lower limit for *minimum* profit from algebra as commonly taught ten years ago; another study by Madsen gives 123 as the median I.Q. of boys (and girls) successful in algebra, and 111 as the median I.Q. of boys who fail; and W. M. Proctor reports that all students with an I.Q. of 120 or over pass their algebra. Professor Thorndike "gives it as his opinion that pupils of I.Q. less than 110 are unable to understand the symbolism, generalizations, and proofs of algebra." [8] At the present time pupils with an I.Q. less than 110 constitute about 60 per cent of the total enrollment in Grade 9,[9] leaving only 40 per cent qualified to study

[7] Reported by Douglass, A. A., *Secondary Education*, pp. 280-293, Houghton Mifflin Company, Boston, 1927.

[8] Reported by Meyers, W. C., *The Mathematics Teacher*, Vol. XXV, No 5, p. 267, May, 1932.

[9] Terman, L. M., *The Intelligence of School Children*, Houghton Mifflin Company, Boston, 1919. See Fig. 10, "The I. Q. Distribution of First Year High School Pupils."

algebra according to this criterion. On strictly utilitarian grounds alone we can justify the study of algebra for only 5 out of these 40. For the other 35, i.e., 87 per cent of all students allegedly competent to study algebra, we must argue the cause of algebra on the basis of appreciatory and disciplinary values.

Presumably far more than 40 per cent of the pupils enrolled in Grade 9 are at present sitting in algebra classes. Do they all belong there? Figures quoted above from the *Biennial Survey of Education of 1922-1924* show us that 75 per cent of all students in high school are in the "academic" courses. The writer has been unable to find corresponding figures for Grade 9 alone, but infers from known rates of elimination in the different grades that at least 60 per cent and probably 65 per cent of all pupils in Grade 9 are in the academic group. If our estimate is correct that only 40 per cent of all ninth grade pupils can study algebra with profit, and if we ascribe no pupils with an I.Q of 110 or over to the non-academic curricula, even so, at least one-third of the pupils in the academic group will get almost nothing from algebra which goes beyond the simplest treatment of formula, graph, and equation. For this third of all students at present enrolled in academic courses, and for the large number of students in commercial, industrial, agricultural, and home economics curricula who will not wish to continue with algebra, we may still be able to save some of the cultural and disciplinary values which algebra can yield, but we shall probably have to resort to a non-algebraic medium to accomplish this.

Algebra as a discipline. Whatever discipline may come from algebra by way of accuracy in reading the printed word, in precise thinking, in checking the reasonableness of results, and from the training in generalization which algebra can give though often it seems not to give it we cannot insist that these outcomes are obtainable from algebra alone, and that without algebra pupils will not be able to develop these qualities. It would seem likely that inspired instruction in English composition, in the social studies, in experimental science, and in the translation of foreign languages could yield all these outcomes. It may be that the very abstractions of algebra point to a greater likelihood of attaining these objectives through algebra than through the other subjects of the school program. Whatever medium we employ for the development of each quality, we must lead the pupil to generalize it, to raise it

out of its local setting, if we wish to insure transfer of the discipline to situations in everyday life. Do we commonly do this? How many of us who cling to the possibility of transfer of training acquired in algebra and other mathematics, regarding it as sufficient apology for our subject and for our teaching it, remember that to convert this possibility into probability much is required of us? Still admitting the possibility of transfer in the most general sense, we feel on surer ground in asserting that our subject develops *mathematical* reasoning, *mathematical* precision, *mathematical* disciplines. Probably we *can* dissociate this precision and this reasoning from their algebraic matrix and transfer them to non-mathematical situations. And perhaps an original mathematical setting is peculiarly favorable for the dissociation and transfer of these and related disciplines. But this dissociation and transfer cannot be relied on to take place without connivance on our part; these are outcomes to be accomplished, not simply expected. There is still a large discrepancy between what we do and what we can do what we might do. We do well to hesitate to press this more general claim of transfer of disciplines inculcated by our subject until we have gone farther toward clearing our title.

A pupil who finds reasoning about mathematical ideas stimulating and enjoyable will get from that exercise much the same sort of thing that another will get from reasoning in another field. Each in his own way will acquire an appreciation of what reasoning is. If we can broaden this appreciation to include an appreciation of what reasoning in general is, so much the better. But there are other things in algebra besides "disciplines" to be appreciated.

Algebra as a social study. Professor Judd has denounced the point of view of those who see only the utilitarian value of arithmetic, who delight to dub it a "tool subject" and so damn it with faint praise.' True, numbers and the number system are tools which we gratefully employ in many different ways, but they are more than tools, as Professor Judd has taken pains to emphasize, and his remarks concerning number in arithmetic can be applied to algebra as well. We wish our pupils not only to appreciate the uses we can make of arithmetic and algebra, but also to understand the very nature of the number system developed by the human race

Judd, Charles H., "The Fallacy of Treating School Subjects as 'Tool Subjects.'" *The Third Yearbook, The National Council of Teachers of Mathematics*, 1928.

over so many ages. This aspect of arithmetic and algebra is indeed a social study.

We teach negative numbers partly because we have a use for them and want our pupils to realize the many applications we can make of them. But utility alone is scant justification for algebra, as we have seen. We also want our pupils to grasp the way in which the extension of our number system to include these new numbers has made possible a generalization of the operation subtraction so that now subtraction is always possible, whereas formerly it was limited by the relative size of minuend and subtrahend. This very striving for generalization is of the essence of mathematics. The student can now look back and appreciate that earlier extension of the number system when fractions were included with an eye to generalizing the operation division. He can understand, further, that these extensions of the number system were not forced upon us by their usefulness, but were suggested rather, and that the decision to extend or not to extend the number system by including new numbers was in each instance essentially arbitrary.

This arbitrariness of our number system and of the laws of operations with numbers is easily demonstrated to a class by developing with them part of the multiplication table for integers in the number scale with base 5. If they demur and snicker at $2 \cdot 2 = 4$, $2 \cdot 3 = 11$, $2 \cdot 4 = 13$, ... $2 \cdot 13 = 31$, they quickly see that these results, obviously bizarre and unorthodox, are just as correct as the more familiar relations. It all depends upon the point of view. There is a philosophy of relativity here that the pupil can easily understand, which is basic to all mathematics, and to fields far removed from mathematics. The extension of the laws of exponents, first announced with regard to positive integral exponents only, is another example of the same idea. Here is matter which can be presented without the aid of algebraic symbolism, though reference to negatives and exponents offers wider exemplification of the principle; matter which can be appreciated also by pupils who are not taking algebra.

To characterize algebra as "mainly a tool subject" is to suggest at once its "mass of meaningless technique." Professor Thorndike pointed out years ago that this much maligned technique can be an important avenue to appreciation. The technique of algebra, the grammar of its symbolic language, is the very essence of the

subject. It is not only not meaningless, it is highly significant. If the technique appears to some students to be devoid of meaning, then the whole subject is meaningless also; they simply do not comprehend what it is all about. To miss this understanding of the true nature of algebra is to miss far more than a bit of "tool technique." That it is far easier to give pupils the relatively unimportant "tool technique" than to lead them to appreciate the real significance of algebra is pointed out above. The latter is the important aspect for most pupils; the former alone—technique without meaning—is useful to no one. Why trouble to bring pupils to this stage if they cannot go on to genuine comprehension?

This insistence on the fundamental importance of a true appreciation of algebraic symbolism, and on the relative unimportance of algebra as a "tool subject" finds support in many quarters. In answer to the question, "For what percentage (estimated) of all pupils enrolled in the ninth grade will algebra—above the elements of formula, graph, and the solution of simple problems by means of one equation in one unknown—be primarily a tool subject?" the median of twenty-five replies was 5. Only six, all teachers in secondary schools, suggested a percentage higher than 20. One answer makes gloomy admission that "few ninth grade pupils get sufficient command of algebra, beyond the topics excepted, to use it as a tool."

The question, "For what percentage of all pupils enrolled in the ninth grade can we justify a further study (then or in a later grade) of algebra as a generalization of arithmetic, in which the laws of multiplication, for example, are extended so as to apply to negative numbers and to irrationals, and in which the nature of the number system of algebra is called frequently to the pupils' attention?" elicited responses varying from 0 to 100 per cent. The median of all replies received from teachers of education (generalists) was 20 per cent; from teachers of mathematics, 60 per cent; from both groups together, about 30 per cent. I quote the following:

An elementary and careful treatment should be included for all who are able to take it, because of the cultural value.

We can justify it for all those who have the intelligence to understand it. Too many later regret not having continued.

Nearly all ninth grade children whom I have met are capable of appreciating more of pure mathematics than we often give them credit for.

These replies are significant when compared with the estimate given above (page 178) that about 40 per cent of all pupils in the ninth grade have sufficient intellectual power to benefit from a substantial course in algebra; and significant also as indicating that students who can appreciate the meaning of algebra ought not in general to be permitted to elect *not* to take it in the ninth grade. Teachers of mathematics in secondary schools were practically unanimous in agreeing that we ought to give our pupils some idea of the development of the number system of algebra. Two or three would limit this to the better students. College teachers of mathematics were for the most part doubtful if it could be done, but were not unwilling to see it tried.

A course in computation. Ninth grade pupils who have had a substantial half year of algebra in Grade 8, have the ability to attain a reasonable facility in the technique of algebra and, more important still, can appreciate the real meaning of algebra, will not need all the ninth grade in which to master simultaneous equations, radicals, and the simplest beginnings of quadratics. Time will remain for the application of algebra to compound interest and related topics, for logarithms, the slide rule, and the numerical trigonometry of the right triangle. These latter topics emphasize computation, whether in regard to arithmetic, to algebra, or to geometry. This return to arithmetic for the consideration of certain advanced topics in the light of the algebra already mastered repeats the idea earlier expressed with regard to Grade 8. It ought also to serve as further answer to those who might feel that the algebra of Grade 8 has been emphasized at the expense of arithmetic.

A non-mathematical course for non-specialists. The course just outlined for the upper 40 per cent of Grade 9 cannot be fitted to the needs of the other 60 per cent by simply modifying certain details. The 60 per cent will have to have something quite different, as they are practically at the end of their mathematical tether. Whatever their vocational objective, they will need to know the common arithmetic of business and the details of modern business practice. Some of this they have learned in Grade 8. The rest comes better as incidental to, and as an integral part of, a course in junior business training not under the control of the department of mathematics and taught presumably by teachers specially trained for that purpose. The bookkeeping which formerly bulked so large

in the commercial curriculum is to-day regarded as the specialized work of only a relatively small number of persons engaged in business and is not a fit topic to be studied by all and sundry who have little aptitude for mathematics. Still more is this true of commercial algebra which is far harder than the ordinary algebra of our secondary schools. There is almost no topic left in mathematics which these inferior pupils can study profitably. If, however, they are unable to grasp the abstractions and generalizations of algebraic symbolism, it may still be possible to school them in abstractions and generalizations mostly non-mathematical, preserving for them in this way some of the values which the study of algebra is supposed to yield to their brighter classmates. If we can devise such a body of instruction, we ought to incorporate in it also as much as possible of the training in logical argumentation which is supposed to result from the study of demonstrative geometry. For presumably this also is a subject which these pupils could not pursue with profit. Let us see what such a course for the group we are considering might contain.

First, it could proceed from simple problems in arithmetic concerning, for example, the varying cost of different numbers of oranges to general questions of the same sort without numbers. Questions like this appear commonly in algebra texts, but we should expect here to get along without the letters *n*, *c*, *x*, inducing the generalization from several numerical instances. From several simple cases all illustrating, for example, the same principle of the law of contracts, agreements to buy or sell, or the shipping of goods on a common carrier, we could expect the pupil to induce the general principle around which each group of illustrations was built. The object here is to provide material for generalizations outside the field of mathematics. Other material can be taken from the sciences where in similar manner the pupil, confronted with a record of observations on the behavior of some animal, or the like, would be expected to induce the underlying general law. The point here is not to encroach upon other school work in scientific or social studies, but rather to collate all sorts of generalizations with an eye to emphasizing the common aspect of so many different situations. If these pupils are not to use the generalized symbols of algebra, they can have, nevertheless, certain significant experiences in generalizing outside mathematics. If the abstractions of algebra are not for them, could they not derive some profit from toying

with the abstractions present in an artificial language of the sort often used in intelligence tests?

They will not learn of the successive extensions of the number system and of the operations with numbers which are part of the spirit of algebra; but without algebra they can widen their understanding of the familiar number system of arithmetic to the base 10 by constructing tables for adding, subtracting, multiplying, and dividing numbers to the base 5, and by solving a few simple problems in this new number system. This is not intended to suggest the topic "scales of notation" in advanced algebra, and is not as hard as it may appear; it merely indicates a sportive adventure in the field of elementary number to bring out the significance of place value in our method of writing numbers. The very incongruity of the number combinations to the base 5 is an important part of the exercise, emphasizing quite dramatically the arbitrary nature of our familiar number relations. Here is a bit of mathematical philosophy within reach of students of average and less than average ability. It is but an arithmetical rendering of one outcome to be expected from a study of algebra.

Most of these pupils will never study demonstrative geometry. Nevertheless, we wish them to acquire, if possible, an appreciation of the nature of a proof based on reasoning; to see the need of assumptions, definitions, and undefined terms behind every body of logic; to distinguish between good and bad arguments; and to be critical of their own, and others', reasoning. Granting that demonstrative geometry lends itself to this purpose peculiarly well, it is not impossible to learn these things from non-mathematical material. An assumption underlies every advertisement, every political slogan; the student can ferret these out, examine them, and see if he agrees with them. We often meet arguments which take for granted the very thing they are trying to prove; the student can learn to detect these cases of "begging the question." We can distinguish necessary and sufficient conditions without resort to mathematics. If a man lives in Ottawa, then he lives in Canada. To live in Ottawa is sufficient to make him a resident of Canada, but not necessary. For living in Toronto is also sufficient to make a person a resident of Canada. On the other hand, living in Canada is a necessary condition for living in either Ottawa or Toronto. Every *If . . . . then . . .* statement is susceptible of these two interpretations: the *if* clause is a sufficient condition for the

then clause, and the *then* clause is a necessary condition for the *if* clause. The converse of a proposition is important in this connection; and here too we do not require a mathematical setting for the consideration of converses and whether or not they are true.

This non-mathematical course for students in the lower 60 per cent of Grade 9 is designed to capture and preserve for these students some of the elementary philosophy of algebra and geometry which every "general reader" ought to get from these subjects -- and also every specialist. This course, like all the mathematics for the upper 40 per cent, can draw appropriate examples from the fields of business, industry, and agriculture. The proposal to diversify our courses in mathematics on the basis of intellectual ability rather than curricula does not mean that we shall now avoid allusion to vocational applications of mathematics, but that these shall be a part of all mathematics for every sort of student so long as he studies mathematics.

Limiting the "regular" course in algebra in Grade 9 to the upper 40 per cent need not be construed as making this algebra a free elective. Presumably those pupils who fall in this group and have passed the algebra of Grade 8 will, with few exceptions, go on to the algebra and geometry of Grades 9 and 10, getting in more extended mathematical form the outcomes which can be attained from the study of these subjects. Those who wish to continue in science or mathematics ought to take the mathematics of Grades 11 and 12. The entire program for the upper 40 per cent should be a coherent and substantial unit throughout the six years of the secondary school, though it is expected that some students will not continue beyond Grades 10 or 11. The work of Grade 7 can be the same for all pupils in the school. For inferior students there should be a separate course in Grades 8 and 9 as indicated above; these students would end their study of mathematics here.

General mathematics and parallel method. It will be observed that at least two subjects in mathematics have been proposed for each grade of the junior high school, and further, that all these are to be considered as parts of a substantial and coherent unit course extending over several years. They are not presented here as forming a course in "fused mathematics," however, since it seems just as important to preserve the essentially distinct characteristics of each subject as to indicate the significant interplay of two or more of them. This point of view is supported almost unanimously

by twenty-eight leading teachers of mathematics in colleges and secondary schools. In answer to the question, "Do you think that algebra and geometry (on the secondary school level) have so much in common that they can be fused, or should you prefer some mode of presentation which reveals significant relations between the two while preserving the distinctive characteristics of each?" all but three preferred the latter. Two of the three answers favoring fusion are given here:

I believe the student finds the logic of geometry sufficiently difficult to make it advisable to proceed some distance in geometry before fusing geometry and algebra, but I would favor fusion at the earliest date at which the student can undertake it without confusion.

I have no objection to "fusion" if it is not used as an excuse for omissions and general weakening of the course, which seems all too common in "fused" courses.

The repudiation of fused mathematics need not mean that we must revert to the practice of devoting an entire year to a given subject, "finishing arithmetic" before we begin algebra, and breaking the continuity of algebra in Grades 9 and 11 with a full year of demonstrative geometry in Grade 10. We can adopt the flexible course of study so common to-day in which the various subjects appear successively, with varying degrees of exposure, all included between the covers of a single textbook under the title *General Mathematics*, or *Eighth Year Mathematics*, or the like. Or we can teach two subjects simultaneously according to the common practice in Europe, where each subject is printed in a book by itself but taught in parallel with a second subject. We commonly say that our teachers in this country are more dependent upon the textbook than are teachers in foreign countries. If the texts we put into the hands of our pupils do not present the essential characteristics of each subject separately, contrast them, and indicate the pertinent cross-relations between subjects, shall we suppose that the pupils learn these things from their teachers and despite the texts? Whether we elect for the present to administer separate and successive doses of the different branches of mathematics, or attempt to present two subjects in parallel, we can at least put into the pupils' hands separate and well-ordered presentations of the different subjects, and then teach as we please. It would seem far simpler to correlate, interpret, and amend two different subjects, given first a clear exposition of each one separately, than to start with a com-

posite of both subjects and then attempt to bring out the essential characteristics of each. We want both correlation and contrast. Do we get both? We recognize individual differences among our pupils; we recognize also a valid social demand for a certain degree of conformity in their behavior. In mathematics, similarly, we want a degree of conformity between the separate subjects, but not such as to submerge their significant individual differences. The parallel method of instruction yields to none in its ability to show the mutual interdependence of the different branches of mathematics; it surpasses all in its ability to preserve the continuity and integrity of the individual subjects.

That there is demand for less confusion and greater continuity of our instruction in mathematics is apparent from the answers received to our questions. These answers reveal complete agreement on the need of greater continuity in the course of study, and unanimous condemnation of the present practice of permitting algebra to become rusty during the fourteen months of forgetting between the ninth and eleventh grades. The parallel method of instruction received equal amounts of approval and disapproval, while a few doubtful persons expressed interest in seeing it tried. This even division of opinion held for college teachers of mathematics, and also for teachers of mathematics in secondary schools. All but two of the teachers of education favored the parallel method. Inasmuch as this question is so evenly contested by those who are closest to it, it seems better to omit from the final record this apparently deciding vote by the "generalists." Even a fifty-fifty response from the specialists on a matter involving so considerable a change in teaching practice, and presented without previous warning or discussion, indicates that there is something here that deserves further investigation.

The consideration of parallel method appeared in two questions widely separated in the questionnaire. This was done to check the replies, but proved to have been unnecessary. On each question the number of persons answering "yes" was one more than the number answering "no"; in all, 19 said "yes," 6 "doubtful," and 17 "no." Some of those answering "no" based their objection on the administrative difficulties to be overcome. The questions referred to read as follows:

1 "Certain advocates of fused or general mathematics have concluded that *demonstrative* geometry stands apart from the rest

of mathematics, resisting fusion. Granting the conclusion, is it desirable to allow the other mathematics to get badly rusted during a fourteen-month holiday while the pupil is learning demonstrative geometry?"

2. "Is there need of greater continuity in the course of study in mathematics?"

3. "Would it be better to teach the demonstative geometry in a more attenuated course through two successive years, paralleling it with instruction (under the same teacher) in the other mathematics which the student ought to have at his command?"

4. "Should you approve an attempt to gain greater continuity of instruction in secondary mathematics by means of parallel instruction in both algebra and geometry in Grades 10 and 11?"

I quote a few of the answers to the questions on parallel method:

> If three years are given to algebra and geometry, we would suggest that the first year be devoted to algebra and that the next two years there be given a course in algebra and one in geometry running simultaneously. [From the department of mathematics of an eastern university]
>
> Yes, we do this. [A very large city high school]
>
> This has been tried but it is an "academic" proposal because every half year our pupils are unscrambled and sent into different classes. Until this nonsense stops, you can do nothing of this sort in a large school. [From the supervisor of mathematics in a large city system]

Evidently the question of continuity of instruction is one that concerns other subjects besides mathematics. Administrative officers have to be careful not to get so enmeshed in the gears of their own machine that they forget that their office is properly the servant of instruction. On the other hand, teachers of mathematics — and doubtless teachers of other subjects also — frequently cloak their own indifference with the mantle of administrative difficulty. This particular difficulty has been easily surmounted in schools where the teachers of mathematics really want a change, as in the Boston Public Latin School, in the Newton High School, and in other schools in and around Boston. Apparently it is not impossible to get teachers to teach algebra two or three consecutive days a week and geometry three or two days, and to assign grades in "mathematics" rather than in algebra and geometry separately. This mode of marking does not mean that the two subjects are "fused" in the teacher's mind, or in the pupil's. The object is to simplify the teacher's records; also, to make it possible to promote

a student who may be deficient in one subject, provided this deficiency is balanced by his showing in the other subject. It is probably wise to omit the algebra during the first few weeks of the demonstrative geometry until the pupils have become familiar with the system peculiar to this new subject.[11] It is obvious that an ordering of the subjects which makes unnecessary a long recapitulation of the algebra of Grade 9 in Grade 11, but proceeds without waste of time to new topics, and which spreads out the instruction in demonstrative geometry over two years so that the geometry may lie in the pupil's mind longer and profit by his own gain in mental maturity, will lend greater continuity and efficiency to our instruction. Perhaps we shall regard this method of parallel instruction with greater favor when we reflect upon the history of the teaching of mathematics in our secondary schools, and consider the forces which have molded our present course of study.

Algebra and geometry were college subjects from 1720 to 1800, but by 1844 Harvard required both these subjects for entrance—a year of each—and by 1860 both subjects were a recognized part of the curriculum in the high schools of Massachusetts. At this time one-quarter of these high schools taught algebra and geometry in parallel during the first two years of the high school course. By 1875 most colleges required both algebra and geometry for entrance and the parallel method had been generally abandoned. The requirement in algebra for college entrance was raised to one and one-half years in the decade 1890-1900 and the Committee of Ten was recommending two full years. This was in 1893. When the colleges required the extra half year, and later the additional full year, it seemed a natural administrative maneuver to tack this on to the course of study then in effect. But the recommendation of the Committee of Ten in 1893 was quite different. They recommended five hours a week of algebra in Grade 9,[12] and "about two hours and a half a week during the two years next succeeding." They also reported, "The conference believes that the study of demonstrative geometry should begin at the end of the first year's study of algebra, and be carried on by the side of algebra for the next two years, occupying about two hours and a half a week." It is evident that they were interested in continuity of instruction. They said further, "If the introductory course in geometry [this referred to informal geometry

[11] See *The Harvard Teachers Record*, April, 1931, p. 80, and April, 1932, p. 84.

[12] They recognized later the possibility of beginning algebra in Grade 8.

in Grades 5, 6, and 7] has been well taught, both plane and solid geometry can be mastered at this time." For science students in Grade 12 they recommended trigonometry and advanced algebra. We shall return to this last recommendation later.

In 1899 the Committee of the National Education Association on College Entrance Requirements met and reported. In order to standardize the concept of a year of algebra and a year of geometry, they defined a "unit" of instruction. This definition has been commonly accepted, while other suggestions of this Committee have been ignored. They recommended *parallel* instruction in geometry and algebra throughout Grades 7, 8, 9, and 10. They called attention to three stages of instruction in geometry: concrete geometry, the introduction to demonstrative geometry, and demonstrative geometry, thus anticipating stages A, B, and C of the British Report on *The Teaching of Geometry in Schools* (1923). They admitted that their schedule made possible a half year of instruction in one subject followed by a half year of instruction in the other, but they expressly disavowed this, saying "that is not to be preferred." They wanted every teacher of mathematics to teach every branch of mathematics. This committee not only expressed its interest in greater continuity of instruction in mathematics; it was definitely out to achieve it!

Geometry in Grades 10 and 11. Despite frequent adverse criticism, demonstrative geometry will probably remain an important part of the course of study in mathematics. This very criticism, however, may cause an acceptance of many modifications in content and method. Just what these modifications will be, is not easy to predict. We have as yet no clear idea of what this subject ought to give those who study it, and no agreement on who should study it. We think we desire different emphases on geometric fact and logic for different sorts of pupils, but the details of this differentiation are still undetermined. When we shall know how much training in mathematical reasoning a pupil can expect to obtain from demonstrative geometry, the extent to which this special training can be extended to apply to reasoning in general—if this be possible at all—and the relative importance of the residual disciplinary and cultural values of this subject, we shall be better able to decide how much of this is suited to students in college preparatory, technical, industrial, agricultural, commercial, and home economics curricula.

The important facts of geometry can be learned apart from the details of logical deduction common to demonstrative geometry. True, the logical connections offer convenient associations of one fact with another that can facilitate this learning, but the mere acquisition of the facts of geometry for some utilitarian purpose can be accomplished without this aid. It can be argued that geometry exemplifies to a marked degree the great advantage of a logical system when one is ordering a multiplicity of facts. In this respect geometry, though not unique, may be preëminent. When we regard geometry from this point of view, however, we are confronted with the delicate task of determining the worth of an appreciative insight and of weighing the possibility of transferring a geometric discipline to the world of thought outside geometry.

There are almost no geometric facts essential to students in college preparatory, technical, industrial, commercial, or home economics curricula which cannot be learned in the junior high school. These comprise the facts concerning parallels, the sum of the angles of a triangle, similar triangles and proportion, the Pythagorean theorem, regular polygons and the circle, areas, volumes, and elementary ideas concerning locus. We have seen, moreover, that the slight divergences in the factual material of geometry essential to the various vocations do not warrant the differentiation of students of demonstrative geometry in Grades 10 and 11 according to curricula. The important outcomes of demonstrative geometry pertain not to facts, but to appreciations and disciplines. Differentiation according to mental ability, therefore, would seem more sensible. Pupils who cannot appreciate an abstract logical system, mathematical reasoning, will find little significance in demonstrative geometry. A course in non-mathematical reasoning and generalization which is better suited to such pupils is outlined above. Let us consider what demonstrative geometry can offer to the brighter pupils—say the upper 50 per cent—in Grades 10 and 11.

There is practical agreement that the main justification of the study of demonstrative geometry relates to logical thinking, at least so far as this is confined to mathematical thinking. The doubts and objections multiply if we assert that the study of geometry improves thinking in general. Almost no one denies that through geometry students can learn to appreciate what is meant by an abstract logical system, especially a geometric system, and can improve their ability to reason logically in mathematical situations. Difference

of opinion exists concerning the possibility of generalizing this ability, dissociating it from geometry, and transferring it to fields outside mathematics. The specialists assert that this dissociation and transfer *are* possible; the generalists—teachers of education—are evenly divided on this point. It will probably be safer to discount the enthusiasm of the specialists here and to interpret the division among the generalists as a "suspended sentence." For whether or not there is possibility of transfer, specialist and generalist alike admit that most teaching of geometry fails in this respect. It will be salutary for teachers of geometry to regard the question as still open, and the final verdict still in their hands. Conceivably they can learn to teach this particular logical system, demonstrative geometry, in such a way as to give their pupils an appreciation of the characteristics common to all logical systems; to insist not only on logical deduction in geometric situations, but to demand also a conscious dissociation of this discipline from its original setting and its application to situations quite outside mathematics. Not until teachers idealize geometry and so reveal it to their students can we expect any appreciable transfer.

That the opinion of general students of education has been correctly represented, and the opinion also of specialists, teachers of mathematics in school and college, will appear from the answers received to a long list of questions concerning geometry, as follows:

1. "It is commonly said that the study of geometry in Grades 7 to 12 should have three main results: (1) a knowledge of important geometric facts, (2) an improved ability to reason logically, (3) an appreciation of an abstract logical system. Do you agree with this statement? If not, what is your opinion?" Twenty-four agreed with this statement; three agreed only in part; and two dissented. One of these last wanted analytic geometry in place of demonstrative geometry. The other qualifications tended to limit any improvement in logical reasoning to purely mathematical situations.

2. "Can the important *facts* of geometry be mastered below the tenth grade?" Twenty-two said "yes"; three said "many of them"; four said "no."

3. "Ought the main outcomes of demonstrative geometry to pertain to logical thinking?" Twenty-seven said "yes"; three said "no."

4. "Ought a course in demonstrative geometry to show the pupils (among other things) the nature of a mathematical system, the

need of undefined terms, the arbitrariness of assumptions, the possibility of arrangements of propositions other than that given in their text? How important is this, relatively?" To the first part of the question twenty-three answered "yes" though four of these were dubious if any but the best pupils could profit by such instruction; six said "no." Most of the doubters were college teachers of mathematics. Teachers in secondary schools maintained on the whole that this was very important. One replied, "Certainly. So important that it's a calamity so few teachers are prepared to give it."

5. "A certain experiment seems to show that demonstrative geometry—as now taught—trains the pupils' ability to reason little or no better than certain commercial subjects. Do you think that demonstrative geometry *can* be so taught that it will develop the power to reason logically more readily than other school subjects?" All but one of the teachers in secondary schools said "yes" to this. Of the teachers of education, five said "yes" and five said "no," while two were uncertain. Some of the replies from general educators were as follows:

> Yes, if that objective is clearly in the mind of the teacher and means are adapted to ends.
>
> I don't know. Perhaps yes, if geometry makes a conscious effort in that direction while other fields continue to be casual about it. If all studies tried to achieve this end, I don't know which would be likely to make the greatest contribution. The social studies, I suspect.

Several replies intimated that for those who like geometry, it is a good medium of logical training; for other students other subjects would be better.

6. "If not [see Question 5], what justification is there for continuing to teach demonstrative geometry?" The answers here were predominantly "none" or "very little"; though some recognized a cultural value in understanding what is meant by a logical science, and some rated it at least as good as other subjects for developing the ability to reason logically.

7. "For what percentage (estimated) of all pupils enrolled in the tenth grade can we justify a protracted study of the logical aspects of geometry?" Four said from 70 to 80 per cent; three said 10 per cent or less; the remainder said 50 per cent or thereabout. An estimate of 50 per cent in answer to this question comports well with our estimate of 40 per cent in Grade 9 who can profit from the study of algebra.

8. "If you still see educational possibilities in demonstrative geometry, do these depend on a certain 'transfer' of the logical training of geometry to situations outside geometry?" In answer to this question seventeen secondary school teachers said "yes," two said "not entirely," and one said that the transfer is "negligible." The teachers of education showed greater divergence of opinion: four said "yes," two "partly," and three "no."

9. "Do teachers of geometry ordinarily teach in such a way as to secure the transfer of those broader attitudes and appreciations which are commonly said to be most easily transferable?" To this twenty-one said "no." Seven others graciously qualified their replies as, for example, "some do," "yes, but in varying degrees," "not as much as they ought." One wrote, "As a rule, no. The pupils' and the teachers' daily aim and concern is not with logical thinking, not with creative, inventive, intuitive thought checked by logical proof, but with 'getting the next two propositions,' 'reviewing the last ten,' or 'doing Exercises 1 to 7.'"

10. "If not [see Question 9], how ought they to modify their ordinary methods in order to secure this transfer?" Some of the answers to this were as follows:

Make every possible effort to supply the common elements of transfer.

They ought to recognize that transfer is not likely to occur unless it is directly sought, and they ought to teach for the purpose of securing transfer. They ought to have clearly in mind the situations to which they are seeking transfer, and make these situations a part of their subject matter.

They ought to employ more illustrations of the use of logic in other fields.

They need to find examples in life where much geometric reasoning can be used and put them before their pupils.

The teaching should be on the thinking level, and not on the mechanical level.

They should illustrate the principles of reasoning in everyday life as well as in geometry.

Bring to consciousness the logical aspects of geometry. Stress the method of analysis. Apply the methods of geometry to other fields of thought.

Our teaching pays a good deal of attention to converse statements and to the indirect proof, laying some emphasis on the form of statement or of argument, rather than on the content. With these exceptions hardly any attention is paid to the [illegible] of formal logic as criteria of judgment. Even in these exceptional cases it is seldom that statements not referring to geometry are ever discussed; in fact the word "converse" here has a sense not permitted outside geometry. This state of things would be much

improved by the addition of one or two logical patterns, such as contrapositive statements and De Morgan's invention of the universe of discourse, and especially by plentiful discussion of examples from non-technical speech.

Require less formal writing of proofs and more thinking.

Stress simple originals with which pupils can succeed, using the theorems chiefly as patterns. Consider analogous steps in well-organized arguments, articles, pleas, etc.

Apply methods to problems in other fields, e.g., economic problems as illustrated in current events.

Insure the dissociation of the method and its application in non-geometric situations. The question then comes whether you are really teaching geometry.

Contradictory as it may seem, I believe that the teaching of demonstrative geometry would be improved if it involved more intuitive material. Some intuitive material from the geometry of three dimensions or conic sections might be included. The introduction of proofs written in essay form but none the less rigorous might be helpful.

These replies show a demand for conscious generalization of the method of demonstrative geometry and its application to other fields of thought. True, this latter is not properly geometry. But what matter, if it makes the whole of demonstrative geometry much more significant? Very likely, a relatively small number of excursions of this sort will suffice to indicate how the logical discipline of geometry can be transferred to situations outside geometry. After all, the human mind can still generalize.

The method of demonstrative geometry unique in secondary mathematics. Evidently the most important characteristic of demonstrative geometry is its method. In this respect it holds a unique position in secondary mathematics. The movement to batter down the partitions between the different branches of mathematics —a commendable movement, be it said, so long as the original contours can still be traced—encountered serious resistance in demonstrative geometry; it would not fuse. The function concept, widely heralded as a unifying principle in all mathematics, applies to demonstrative geometry in a sense quite different from that in which it is applied to arithmetic, algebra, and factual geometry. For these latter it means the dependence of one variable upon another; for demonstrative geometry it indicates, if anything, the dependence of conclusion upon hypothesis. True, the numbers of arithmetic and algebra are employed in geometry and the logical system of geometry has its counterpart in arithmetic and algebra,

but both are on a higher level than we ordinarily attain in our present instruction in secondary schools. The student should appreciate the relation between irrational numbers and incommensurables in geometry; and he should know that, like geometry, arithmetic and algebra are abstract logical systems—so abstract, in fact, that for exemplification of a logical system in detail, we prefer the concrete configurations of geometry to the abstractions of arithmetic and algebra. We can, indeed, show him very little of algebra viewed as a postulational system, hardly more than the part which definitions play in topics like negative number and fractional exponents. We prefer, on the whole, to illustrate the logical method of geometry by pertinent reference to fields outside mathematics. To facilitate this transfer we ought indeed occasionally to abandon the usual columnar arrangement of the written proofs, in which the statements *Therefore* . . . appear always on the left and the reasons *because* . . . are always on the right, in order that the student may practice the presentation of logical argument in ordinary essay form such as we use in everyday life.

The method of demonstrative geometry is not limited to detached arguments. It reveals itself further in extensive and coherent logical systems wherein we recognize the need of undefined terms, definitions, and assumptions. We see that theorems in any system are true only in relation to the initial assumptions; if we vary the assumptions we have a new geometry. We can do this easily within the field of Euclidean geometry by substituting for the parallel postulate of the textbook any one of the theorems on parallels immediately following it. We can remind our pupils that textbook writers frequently construct variant geometries in this way, and can show them other texts which differ from theirs in choice of assumptions and in sequence of propositions. The student will appreciate the meaning of a logical system far more from this insight into the possibility of several, many, geometries than from the dogged mastery of "one absolute geometry."

The logical chain of the geometry text used is probably not complete. No matter, so long as the student sees where the breaks occur. Every geometry for adolescents permits such gaps in its logical framework in order to keep within the pupil's power of comprehension. This is perfectly proper. It would be highly instructive, however, if the text called the student's attention to each such break in the reasoning.

This emphasis on the logical aspect of demonstrative geometry indicates the desirability of reëxamining the introductory chapter in current texts. It is not unusual to find diagrammed therein certain interesting optical illusions to show that we cannot always trust our eyes, but must check our observations by other means. The authors do not suggest that a more exact experimental method would suffice; their aim is to persuade the student of the need of logical development in geometry. Would it not be more significant to supplement this with excerpts from our everyday reasoning, and show that we need a check not only on our eyes but also on our logic?

A class in geometry will understand that the assumptions must all be consistent, but need not be independent. They may be few or they may be many. If we start with a brief list of independent assumptions, we shall have to prove many apparently obvious propositions by long chains of involved and difficult reasoning. We can avoid this complication by taking these obvious propositions as assumptions. In this way also we can avoid all mention of superposition, which, while logically defensible, is a psychological stumblingblock at the beginning of the course and not in keeping with the educational aim of the instruction. Present practice in this country takes "motion" and "rigid" as *undefined*, *postulates* rigid motions of geometric figures, *defines* "equal," and *proves* equality by rigid motions, i.e., by superposition. The British Report on *The Teaching of Geometry in Schools* (1923) suggests that it would be simpler to take "equal" as *undefined*, *postulate* the existence of equal figures, i.e., the uniformity of space and the possibility of reproducing figures, and to use this postulate of equality instead of a postulate of superposition to *prove* equality. In the same spirit, it seems to the writer that it would be still simpler to take "equal" as *undefined*, *postulate* outright the first proposition concerning equal triangles, the case of two sides and the included angle, and *prove* by this means the second proposition on triangles, the case of one side and the adjacent angles—or even to postulate all three propositions on equal triangles, and one or two others also, at the beginning of demonstrative geometry.[13]

An allusion is made above to the desirability of linking the in-

[13] A more detailed account appears in an article by the author, "Notes on the First Year of Demonstrative Geometry in Secondary Schools," *The Mathematics Teacher*, Vol. XXIV, No. 4, pp. 213-222, April, 1931.

commensurable case with irrational number, as much to show the full significance of irrationals as to complement the proofs of commensurable cases in geometry. It happens that our habit acquired in arithmetic of approximating an irrational number by means of trial-and-error sifting of rationals into two groups—too small, too large—is essentially the definition of the irrational. Incommensurables are defined similarly in terms of commensurables. If MN is parallel to side BC of triangle ABC, and AM is not commensurable with AB, we can approximate to AM (and so define it) by sorting out the commensurable lengths Am', Am'', Am''', . . . which are too small, from the commensurable lengths AM', AM'', AM''', . . . which are too large. Parallels to BC through m', m'', m''', . . . and M', M'', M''', . . . meet AC at n', n'', n''', . . . and N', N'', N''', . . . The proof of the commensurable case tells us that $An' = k \cdot Am'$ . . . ; $AN' = k \cdot AM'$, ... and $AC = k \cdot AB$. These commensurables are divided similarly into two groups—too small, too large—which together define the incommensurable AN and state that it is equal to $k \cdot AM$. Teachers can impart the spirit of this, if not the details, to their classes in order that they may better appreciate the nature of the number system and its relation to geometry. It is not expected that the students will memorize an official paraphrase of this "to please teacher" or to satisfy an examination requirement.

The relation of number to geometry is fundamental and students should appreciate this, but this appreciation must not be won at the expense of an understanding of the logical outcomes of demonstrative geometry, which it is almost unanimously agreed supersedes all else. Pertinent references to trigonometry and analytic geometry are illuminating and proper so long as they do not hinder the logical development of the demonstrative geometry. Other material suitable for illustration and emendation of present subject matter is to be found in the geometry of three dimensions and in certain topics of "modern geometry" such as duality, similitude, power of a point, projection, and inversion; though here, too, it is important that the embellishments be not allowed to obscure the main theme, but be admitted only to give it greater significance.

The ideas expressed above concerning a longer list of assumptions, some mention of incommensurables, and the subordination of numerical trigonometry to the logical outcomes of geometry receive support from leading teachers of mathematics in colleges and secon-

dary schools. In reply to one question, "Inasmuch as the method of superposition is at variance with the usual logical tenor of the text, should you object to postulating the congruence theorems instead of proving them? And the theorems concerning central angles and their corresponding arcs likewise?" nineteen expressed themselves in favor of postulating the theorems mentioned, and eight were opposed. Five of these eight were college teachers of mathematics, and their opinions were exactly balanced by the opinions of five other college teachers of mathematics. The teachers in secondary schools were overwhelmingly in favor of extending the list of postulates.

In reply to the question, "Are you content to see incommensurables commonly disregarded? Should you favor a treatment of incommensurables which depended on the pupil's earlier experience with irrational numbers instead of on a 'theory of limits'?" six said they were content to see incommensurables disregarded. Three of these taught mathematics in secondary schools, and three in colleges. Eighteen wished to see some treatment of incommensurables retained, three of these favoring a treatment in terms of limits.

When asked, "Would you interrupt the logical development of demonstrative geometry with a treatment of numerical trigonometry in connection with similar triangles, or should you prefer that this topic had been covered earlier in intuitive geometry or algebra?" eighteen replied that they preferred to have the trigonometry merely incidental in the course in demonstrative geometry, two were uncertain, and five were willing to have the trigonometry play a large part. Four of these five were teachers in secondary schools. Two of them said, "The interruption is pleasant." A third desired "the correlation of number with geometric magnitude for expanding the number concept and for clarifying the fundamental ideas of arithmetic, and the utilization of algebraic and trigonometric ideas wherever they will serve either for computation of geometric magnitudes or for demonstration of theorems." The opinion of the preponderant eighteen indicated that they favored earlier emphasis of the numerical trigonometry before the student should have begun demonstrative geometry; that they would admit it gladly for the "computation of geometric magnitudes," but would not use it for the "demonstration of theorems." This opinion was well expressed by a college teacher of mathematics who wrote, "I do not believe in 'covering' a topic and leaving it. I am in favor of 'touching on' an

important topic at every opportunity. I prefer that numerical trigonometry be begun in algebra and carried along in cumulative reviews throughout geometry."

Certain leading teachers of secondary mathematics, almost all of whom are engaged also in the training of teachers, were asked, "How would you link the idea of functionality with the logical abstractions of geometry?" The replies were, respectively:

I do not know.

I'd like to know.

I wouldn't.

Until I have a better, or different, conception of functionality, I would say that this is the wrong place to attempt the formulation of such a link.

. . . through continuity and variation.

. . . through the measurement of angle and the comparison of areas.

. . . through Cartesian geometry.

Every proof shows that the outcome depends entirely on the data. Sometimes a slight change causes a complete change in the results.

It will be noted that these last four replies link functionality with certain numerical relations in geometry and not with its logical abstractions; with geometry, but not with demonstrative geometry. The logical method of demonstrative geometry is undoubtedly unique in secondary mathematics. Let us try to preserve its integrity and widen its applicability.

This insistence on the importance of the logical aspect of demonstrative geometry as contrasted with the facts of geometry is significant when we consider the sort of geometry, if any, we should give to students in the different curricula. It seems clear that, regardless of the particular curriculum a student may elect, he should have the opportunity to study and appreciate the logical framework of demonstrative geometry if he has the requisite mental ability, and should be spared prolonged effort in this direction if he has not. This appears from the answers of teachers of mathematics and college teachers of education to a series of eight questions asking them whether geometry ought to be taught in the tenth, eleventh, and twelfth grades in technical curricula, in industrial curricula, in commercial curricula, in home economics curricula; and whether the geometry of these different curricula ought to stress the logical or the factual outcomes. Only a very few would not offer geometry to students in commercial or home economics curricula. Almost all preferred that it be an elective for students in these curricula and that the subject matter stress the logical as well as the factual as-

pects of the subject. For students in industrial curricula there was a much greater demand that the geometry be obligatory, and a corresponding lessening of the demand for the logical outcomes; for those in technical curricula an even greater insistence on geometry as a prescribed subject, and on the logical outcomes also. Except for the students in technical curricula, there was no disposition to prescribe the logical rigors of demonstrative geometry for those who cannot endure them. For these and all others it is clear that geometry as an elective was not deemed worth while unless its logical side be stressed along with the geometric facts.

A comprehensive course in mathematics for Grade 12. In the course of study in effect in most schools to-day, the mathematics of the twelfth grade consists of solid geometry and trigonometry. Greater continuity in the course of study could result in a somewhat earlier completion of algebra and geometry, coupled with greater comprehension of these subjects, and would make possible an earlier start on the advanced subjects. Many schools, especially those which have a six-year course in secondary mathematics beginning with the seventh grade, can give a year and a half to advanced mathematics. Let us assume, however, that we have but the twelfth grade available for the advanced subjects and consider whether we make the best use of this time.

The solid geometry gives valuable training in the visualization and representation of figures in three dimensions; it also provides further exemplification of the logical methods learned in the demonstrative geometry of two dimensions. In the logical phases, however, we tend to be less exacting than in plane geometry, and the difficulties of our pupils are not in this quarter, but are concerned rather with the actual relations between lines and planes in space. This reflects, perhaps, our opinion that we have reached a point of diminishing returns from the sustained logical treatment of geometry and ought to concentrate on other matters, such as the facts of solid geometry and the further development of the pupil's spatial imagination. Important as these latter are, we ought to recognize that a large part of this training in visualization and representation of three-dimensional figures is more appropriate to the lower grades of the secondary school, although some of it, doubtless, is difficult enough to warrant its position in the twelfth grade. It is not clear, however, that because part of solid geometry is difficult, we ought to withhold all of it until the final year of the secondary course. A

great many of the facts of solid geometry are within the reach of students in the junior high school and ought not to be denied those who go no farther. There is less reason to continue giving a half year of solid geometry in the twelfth grade when we discover that the colleges, while interested that solid geometry receive its proper share of attention in the secondary school and willing to credit a half year of it for admission, are even more interested in other branches of mathematics. The higher technical schools and the scientific schools of universities find that their students' difficulties in freshman mathematics are due mainly to feeble and fumbling technique in elementary algebra rather than to any lack of information about the geometry of three dimensions. They are not particularly interested in the topics usually covered in advanced algebra, but they definitely desire that their students give evidence of greater algebraic experience and maturity, and recognize that this would undoubtedly result from further work of any sort in algebra. They all demand trigonometry and would regret any abbreviation of the present course. Some of this firm stand in favor of trigonometry derives from their fundamental interest in algebra, for in the trigonometric equations and identities they see excellent opportunity for further exercise of algebraic technique, almost the only opportunity for students who offer solid geometry and trigonometry for admission. Without yielding the formulation of the course of study in secondary schools to the demands of colleges and higher technical schools, we can at least study the latter's desires and opinions to see if they contain suggestions of value. These institutions are no longer inclined to dictate to secondary schools, even though they are not wholly satisfied with their products. Undoubtedly they would be quite content to leave the choice of topics in mathematics entirely in the hands of the secondary schools and to build their courses on top of this foundation, if only they had assurance that the secondary course in mathematics was substantial, thorough, and continuous from start to finish, and in the hands of competent teachers. Taking their suggestions, and with them the suggestions also of teachers in secondary schools and of teachers of education in general, let us try to devise a course for Grade 12 which shall be for the best students an appropriate summary and final goal of their entire course in mathematics in the secondary school.

Trigonometry. In a course of study which is thorough and continuous, a full half year of trigonometry in addition to the logarith-

mic solution of the right triangle now included in elementary algebra seems hardly defensible. If a half year was sufficient before this change was made, something like two-fifths of a year, say fourteen weeks, ought to be ample for the remaining topics, i.e., the oblique triangle and the analytic trigonometry of identity and equation. This suggestion implies no lessening of the present requirement in trigonometry, but merely limits to a half year the aggregate of all time spent on it.

Solid geometry. Offsetting somewhat the argument given above in favor of reducing the amount of time devoted to solid geometry in Grade 12, is the contention that in this country in school and college alike we tend to neglect the mathematics of three dimensions and should do nothing to augment that tendency. The writer would not sacrifice the solid geometry cut out of Grade 12, but would include it in the intuitional geometry of the earlier grades. The material thus demoted would be principally the mensurational propositions. For quasi-logical treatment in Grade 12 the writer would reserve most of Book VI, and selected theorems from the later Books, especially those concerning the sphere. The term "quasi-logical" is used because the course would not be confined to a logical treatment of the subject matter, but would include also the elaboration of certain configurations with careful use of drawing instruments, and computations based thereon involving numerical trigonometry. The appendix of the *College Entrance Examination Board Document*, No. 108, which defines the requirement in geometry, gives further details with many diagrams. This was prepared by Professor W. R. Longley of Yale University and is the source of the suggestion made above. The time devoted to solid geometry under this suggestion would be in the neighborhood of ten weeks.

Algebra and analytic geometry. The desire of the colleges for surer technique in the elementary operations of algebra is reminiscent of the familiar plaint of business men with respect to the fundamental operations in arithmetic. They would gladly sacrifice the delightful excursions beyond the field of arithmetic with which the schools beguile their future employees, if only they could count on greater expertness in the fundamental operations of arithmetic. The schools must meet this challenge, but not necessarily by confining the necessary drill to arithmetic. They must provide for sufficient overlearning to leave the desired residue of skill, but some

of this overlearning can profitably be acquired in fields different from "the same old arithmetic," though closely related to it. Similarly, we can meet the request of the colleges for greater mastery of algebraic technique—if we decide that it is for the best interest of the student to do so—by other means than by mere repetition of the elementary operations of algebra in their most obvious and familiar form. Perhaps we can enhance the drill by providing the necessary repetitions in a variety of novel settings, in topics taken from advanced algebra, or in an extension of the earlier work in graphs to include the analytic geometry of straight line, circle, and parabola.

Earlier in this article allusion was made to the algebra of business, commercial algebra, as less appropriate for students in the commercial curricula of high schools than for those students who intend to go to college before entering business; appropriate in some degree for all, however, to the extent that all of us have some contact with the business world and ought to understand the important part which compound interest plays in simple situations involving sinking funds, annuities, buying on the installment plan, and the operations of coöperative banks. This sort of algebra makes significant application of the binomial theorem and offers ample opportunity for drill in fractions and exponents. Some of it belongs in the eleventh grade as appropriate illustration of the binomial theorem. Further details can be given in the twelfth grade if desired.

Of all these suggestions—advanced algebra, analytic geometry, commercial algebra—the analytic geometry offers probably the most natural point of departure. Five weeks ought to suffice for significant treatment of straight line, circle, and parabola, with sufficient exemplification of the solution of "geometry originals" by analytic methods to show what it is all about. Theorems concerning parallelograms, trapezoids, and concurrent lines of a triangle require very little analytic geometry beyond that already taught in algebra to-day, and reveal the method of analytic geometry very effectively. One of the biggest obstacles to overcome in the beginning of this subject is the prejudice of the pupil in favor of positive numbers. It is hard for him to recognize the point (a, o) as representing points on the negative axis of x equally with points on the positive axis. If the analytic geometry teaches him nothing else, it will have served to drive home this fundamental idea of the generalized num-

ber system of algebra. In addition, this subject will undoubtedly give him much practice with equations and fractions of a sort to extend still further his appreciation of the symbolism of algebra and the technique of employing it.

The theory of equations and the derivative. The foregoing demands nothing beyond the linear and quadratic equations of elementary algebra. Since pupils generally consider the approximate determination of roots of equations of any degree higher than the second as quite beyond their powers, it would seem highly desirable to disabuse them of this idea, to explain that in general the approximate methods are the only methods for equations of the third degree or higher, and to show them that their knowledge of graphs and functions is not restricted to quadratics. In addition to the analytic geometry, the writer would suggest, therefore, a brief treatment of the theory of equations which would include the remainder and factor theorems; the determination of rational roots of polynomials; Descartes's rule of signs without proof; the proof that complex roots occur in pairs; and as much of differentiation as is necessary to get the equation of a tangent to a polynomial, to determine relative maxima and minima, and to locate irrational roots by Newton's method. The allotment of not over six weeks to these topics indicates the scope of the suggestion.

It is frequently asserted that secondary mathematics is not really complete until the essential spirit of the differential and integral calculus has been revealed. It is easier to subscribe to the ideal than to realize it in practice under present conditions in the schools, where relatively few teachers are equipped to teach this subject. Nevertheless, certain schools now offer some bits of the differential and integral calculus and have for several years. If it seems wise to encourage this movement, teachers will learn how to bear their part in it. This is the time to consider all proposals for a comprehensive and climactic course for the twelfth grade, even if we decide later to qualify certain more novel recommendations with the proviso that they wait upon the competence of teachers to impart them. The topics from the differential calculus suggested above to accompany the treatment of certain topics in the theory of equations represent about the usual recommendation for this subject and the usual point of application. The integral calculus is less frequently mentioned. Still, if we hold to our ideal that secondary mathematics will not be fully rounded until we have revealed the

spirit of the integral calculus, we must set down here the hope that sometime we can follow this brief treatment of differentiation with a significant exposition of the inverse process, using the indefinite integral to determine the areas under curves and the volumes of solids of revolution.

A comprehensive examination. This proposed course for Grade 12 can be briefly summarized:

Completion of the present course in trigonometry (14 weeks).

Solid geometry—logic, drawing, and incidental trigonometric computation (10 weeks).

Further work in algebra, including the analytic geometry of straight line, circle, and parabola (5 weeks); including also some basic ideas on the theory of equations with pertinent application of the derivative (6 weeks).

A *comprehensive examination* based on the work of these 35 weeks.

This examination will be at the same time a general examination on the entire course of study in mathematics in the secondary school.

Here we have a comprehensive course of study in advanced mathematics which preserves the essential unity of the different branches, while showing important relations between them. The analytic trigonometry demands facility in algebra; the mensuration of solids is designed to involve numerical trigonometry; the analytic geometry applies algebra to geometry; the theory of equations has a geometric setting and leads to an introduction to the methods of the calculus. An examination covering a year's course of this nature would serve at the same time as a general examination on the entire course of study in mathematics throughout the secondary school, and would be in itself a significant educational experience for the student, who would find therein a fitting climax for his mathematical studies. Those who do not wish to elect this final year of advanced mathematics would find similarly that an examination at the end of the eleventh grade in the algebra and geometry which they have been studying throughout the year would be at the same time a comprehensive examination on all their course in secondary mathematics.

The proposal of a course of study in advanced mathematics which ignores the present division into half-year units, and the further proposal of a comprehensive examination at the end of the twelfth grade, find support in many quarters, as we shall see. The so-called comprehensive examination in advanced mathematics (CpH) of the College Entrance Examination Board would seem to

make progress in this direction difficult, though surely not impossible, for the Board has always been ready to give careful consideration to new proposals and to establish alterna ive requirements whenever there seemed to be a demand for them. However, this CpH examination cannot be regarded as truly comprehensive at all, and school and college would both be better served by an indivisible examination covering a year's work in all three advanced subjects—trigonometry, solid geometry, and advanced algebra. Under advanced algebra are included the analytic geometry and calculus mentioned above. On such an examination it would be possible to set questions involving two or more advanced topics simultaneously. Then for the first time we should have a truly comprehensive examination in advanced mathematics. The present requirement governing this CpH examination is a development of the original regulation establishing the New Plan of admission to college, under which a candidate offers four subjects and is examined up to the limit of the instruction he has received in each. To follow precisely the original intent of this regulation would mean the preparation of a comprehensive examination in elementary mathematics, as at present, and seven other comprehensive examinations involving elementary algebra, plane geometry, and each combination of the advanced subjects taken one, two, or three at a time. For several years these seven were combined in one paper (Cp4) from which the candidate selected questions according to the extent of his preparation. Obviously no one of the examinations contained in this composite could fairly be called comprehensive, partly because of the time limit imposed, but principally because no question on the elementary topics could contain reference to any advanced topic, and no question on an advanced topic could depend on either of the other advanced topics, for fear of injustice to one or another of the seven different groups of candidates. The suppression of this examination in favor of the CpH, and the restriction of this latter to those candidates who have studied at least two advanced subjects, corrects certain minor abuses of the examination system, but still offers a paper which attempts to examine candidates prepared in a multiplicity of different combinations of advanced subjects, although now reduced from seven to four. If the next adjustment could subtract another three and leave us with but one combination, we should have a genuine comprehensive examination in advanced mathematics, and should satisfy the original intent of the New Plan.

Such an adjustment requires an agreement between colleges and secondary schools on an indivisible course in advanced mathematics, and an examination framed in accordance with that agreement. We have had similar agreements heretofore; it is not too much to expect others.

Opinions on the proposed course for Grade 12. Let us see to what extent teachers of mathematics in schools and colleges, and teachers of education in general, support the ideas on a course for Grade 12 as expressed above. In response to the first part of the question, "Would the mathematical training of high school seniors be more effectively capped by a composite course in Grade 12 which should include Book VI of solid geometry, a numerical and somewhat trigonometric treatment of Book VII, the remainder of numerical trigonometry and analytic trigonometry as at present, and the theory of equations from advanced algebra; or is the present practice of selecting two of the three advanced topics and ignoring the third about the best we can hope for?" eighteen teachers of mathematics said "yes," six (four of them college teachers) said "no," and one was doubtful.[11] In response to the question, "Should you wish to encourage *competent* teachers to make some use of differentiation in connection with the theory of equations in Grade 12 and of the indefinite integral for the mensuration of certain solids?" nineteen teachers of mathematics said "yes," six (three of them college teachers) said "no," and two (both college teachers) were doubtful, preferring not to pyramid novelty upon novelty. The question, "Do you favor a comprehensive examination at the end of the twelfth grade for students who have undertaken an extensive program in mathematics?" received the assent of almost all the teachers of mathematics in secondary schools and of teachers of education. Only two said "no," and only two were undecided. One who favored the comprehensive examination added, however, that he had never seen a published comprehensive examination that would be suitable for the purpose. Several approved the comprehensive examination, but wanted cumulative reviews and cumulative examinations throughout the entire secondary course.

The following opinions concerning the comprehensive advanced course for Grade 12 show the extent of the difference of opinion

[11] It should be noted that this question does not entirely agree with the course suggested above for Grade 12; it serves, however, to indicate the attitude of teachers on this general problem.

among teachers of mathematics; in most cases this difference is more apparent than real.

One teacher who favored the composite course, including a bit of differential calculus, asked:

But why omit Book VIII? Spherical geometry with its surprises and its interest seems to be left out all around and yet I think it is thrilling.

An excellent answer to this and to other similar protests is not to omit spherical geometry, and some of it has been included in the course outlined above. One of the few dissenters replied:

I believe we shall have better results, certainly for the student going to college, if we do not undertake more than two of the advanced topics. I am convinced that time would not permit anything but the most superficial acquaintance of the student with the two concepts (differentiation and integration) if the necessary work to cover the usual ground is not to suffer.

It is part of the plan that the "usual ground" will not be covered, but that certain topics will be omitted to make room for the new topics. The danger of a superficial treatment is real and so must be avoided, here as elsewhere. The beginning of any new venture runs the risk of an initial stage of superficiality, but it is not intended that this shall be a permanent, or even temporary, feature of this new course.

A member of the division of mathematics at Harvard wrote:

I have often wondered why algebra was left out of the advanced course in mathematics in secondary schools. It seems to me unfortunate, and I heartily endorse the movement to replace some of the demonstrations in solid geometry by selected topics from algebra. The theory of equations is an excellent choice. Determinants and their application to systems of linear equations would be next in line. The minor topics sometimes treated are hardly worth while. I am not quite so enthusiastic about the introduction of a little calculus, at least at the present time, for two reasons [15]—first, briefly, to avoid trying too many new things at once; second, lest the later treatment of the calculus in college lose some of its savor.

A group of five members of the department of mathematics at Princeton replied:

Because of the great importance of algebra in mathematics in college and the relative unimportance of solid geometry, we advocate a combination of trigonometry and advanced algebra and have been doing so for at least five years. We favor the use of differentiation of the polynomial for abler

[15] From this point to the end of the paragraph the wording of the answer was summarized by the writer.

students but feel that the indefinite integral should be postponed until college.

A member of the department of mathematics at Yale wrote:

This idea (of a composite course) appeals to me. General agreement on details may be difficult to get. I should spend little time on Book VI. I should prove nothing that is self-evident as a fact. I should emphasize the trigonometric treatment of Book VII because of its value in developing space perception. I should begin the year with analytic trigonometry and end with solid geometry as offering the best climax to a cumulative course.

He approved the inclusion of certain topics from the calculus.

The consensus of members of the department of mathematics at the Massachusetts Institute of Technology seemed to be that they would be willing to accept a comprehensive examination in advanced mathematics in place of separate examinations in solid geometry and trigonometry, provided this new examination should demand a greater competence in algebra without lessening the present requirement in trigonometry. They were not deeply interested in solid geometry and would gladly forego some of that in favor of greater competence in algebra, even elementary algebra, with extended application to straight line and circle in analytic geometry and with a little, perhaps, of the theory of equations. They did not care for determinants, but attached the highest importance to the mastery of analytic trigonometry. They were interested in encouraging an introduction to the calculus in secondary schools, but not very much interested; and they would surely wish the teachers who undertake this work to be competent to do so.

From another member of the division of mathematics at Harvard the following reply was received:

What is needed is drill, or practice, in correct algebraic manipulation. When our students fall short in algebra (I mean college students), it is because they cannot add two fractions correctly, or cannot handle exponents, or the like. It isn't because of theorems they don't know, like the binomial theorem, or the relation between coefficients of an equation and the symmetric functions of the roots. Their big need is accuracy in algebraic transformations. This defect, i.e. innocence of manipulative ability, has a first-rate chance in trigonometry, i.e. in analytic trigonometry. There you get transformations galore, with algebra and trigonometry working together.

Another field for excellent practice in algebra is analytic geometry. I should prefer putting in some of that to putting in theory of equations. It hooks up immediately with the trigonometry and the solid geometry, and gives a better introduction to the ideas of the calculus. This connects also

with the pupils' graphs in algebra, and moreover, it has an appeal, namely of a machine that works. Finally, analytic geometry helps much to create an appetite for theory of equations at a later date.

Of course I'm all for the schools trying to frame an ideal education from their standpoint, and for the colleges taking what they get (or the best of it). Precise topics are important almost not at all. Maturity is the big thing. When colleges learn this, we shall have better education. But that's all another story.

Certain teachers of mathematics in secondary schools replied as follows:

I should like to see some advanced algebra or analytics replace a part of solid geometry. I should favor a brief study of the elements of differential and integral calculus, but not as indicated in the question.

It is my opinion that whatever subject is taught should receive thorough treatment. Preserve us from a "hodgepodge." Not only the teachers, but the students also must be competent. I teach as much differentiation as they can stand and use in the theory of equations. I have not taught the indefinite integral.

I would have a *minimum* of solid geometry, trigonometry, analytic geometry, and calculus. I have given such a course this year to superior students.

A recent syllabus for a half-year course in advanced algebra now effective for the State of New York includes the analytic geometry of straight line and parabola, and an optional section on differentiation with applications to problems in maxima and minima, velocity, and acceleration.

From this diversity of opinion a body of ideas emerges on which almost all can agree. First, if we devote Grade 12 to but two advanced subjects in mathematics, the two we regard as most important, we are compelled to ignore other advanced topics some parts of which we wish to retain. Second, we ought to provide a significant, substantial, composite course in advanced mathematics as a fitting climax for the entire course of study in secondary mathematics. Third, we had better not tamper with trigonometry. Fourth, the amount of solid geometry can be reduced, and demonstrative methods be replaced in large measure by an intuitive and computative treatment. Fifth, those who are to use mathematics in college ought to have greater proficiency in algebra. One way to obtain this is through analytic geometry. Sixth, there is considerable interest in certain topics of the calculus as a proper closing theme for the course of study in secondary mathematics, in order that the student may glimpse the methods of

modern mathematics and the sort of problems it can solve. Seventh, this program should culminate in a genuine comprehensive examination.

Summary. This article collates the theories of general educators with the theory and practice of teachers of mathematics, and shows a surprising agreement among them. All are interested to strengthen the pulse of junior high school mathematics, to provide the proper course for inferior students as far as they can pursue mathematics with profit, to give greater substance and coherence to the mathematics program for superior students throughout the entire six years of the secondary school, to permit students to terminate their mathematical studies at any one of several stages, while not interrupting the continuous advance of those who intend to persevere to the end; in short, to recognize individual differences and diversify the offering accordingly; to recognize also the need of individual development and meet this need with a coherent and continuous program.

The practical application of these findings to the course of study can be represented schematically as follows:

For Superior Students	For Inferior Students
Grade 7	
Geometry (a full half year) and arithmetic.	Geometry (a full half year) and arithmetic.
Grade 8	
Algebra (a full half year) and arithmetic.	Formula, graph, equation; arithmetic; geometry including trigonometry of the right triangle.
Grade 9	
Algebra, advanced arithmetic (computation), and numerical trigonometry—for the upper 40 per cent of students in this grade.	Junior business training, with incidental arithmetic, taught by the commercial department. (This ends the prescribed course in mathematics for the lower 60 per cent.) A half-year non-mathematical course in generalization and logical argumentation designed to preserve so far as possible the outcomes of algebra and geometry.

Grade 10

Demonstrative geometry for the upper 50 per cent of students in this grade. After this is well started, a continuation of algebra in parallel with it. (Those who take algebra in Grade 9 ought to continue through Grade 10 before dropping mathematics.)

No mathematics above this point for inferior students.

Grade 11

Demonstrative geometry and algebra in parallel, completing the usual course in elementary mathematics.

No mathematics.

Grade 12

A comprehensive course in trigonometry, solid geometry, algebra, analytic geometry, and some reference to the methods of the calculus.

No mathematics.

The differentiation in subject matter is made according to mental ability and not according to curricula. Mathematics deals with abstractions and also with the practical application of these abstractions to life. Every student ought to get as much as possible of both phases. The mathematics of the shop is little different from mathematics in general. A course in shop mathematics serves mainly to capitalize the shop interest of the pupil; it ought to turn his interest to mathematical abstractions and generalizations, and extend these beyond the field of mathematics, just as every good course in mathematics strives to do.

The secondary schools copied the elective system and "credit counting" from the colleges. Now that the latter are turning from this to the balancing of wide distribution with concentrated mastery of a chosen subject, the secondary schools might consider if they too cannot provide greater continuity of instruction and substantial mastery of a subject for students. As an aid toward that end, parallel instruction has been proposed in algebra and geometry, followed by a comprehensive course in advanced mathematics for the best students, and capped by a comprehensive examination on the subject matter of secondary mathematics. Such a course pre-

serves the essential individuality of each subject and reveals also the important relations between subjects. Each subject is viewed as a separate whole, and as part of a larger whole.

This program demands more of the individual teacher than is now expected. In fact, certain adverse criticisms of this program were directed not so much at the ideas proposed as at the apparent failure to recognize how foreign they are to the professional equipment of most teachers in service. The innovations proposed come at significant points in the course where a little, deftly administered, will go a long way. These many "mickles" make a mighty "muckle." An understanding of the number system of algebra, the relation of irrational to incommensurable, an appreciation of what an abstract logical system really is, are all matters within the grasp of teachers in our schools and are a necessary part of their professional equipment. We have paid glowing tribute to the work of Professor J. W. Young in behalf of secondary mathematics in this country. For a real appreciation of his contribution, let us read again, and study carefully, his book on *The Fundamental Concepts of Algebra and Geometry*. With that as background let us return to our classes, not to teach the matters set forth in his book, but ready always with the point of view which, by a brief sentence here and there, can orient the pupil and give him a genuine appreciation of the subject he is studying. This program demands also the sympathetic oversight of a supervisor who is interested in both bright and dull pupils, is thoroughly familiar with the mathematics of all six grades, and can teach in every one.

The proposals made in this chapter aim to reconcile two philosophies: one, the philosophy of the Committee of Ten (1893) and of the Committee of the National Education Association on College Entrance Requirements (1899) favoring a substantial, continuous course of study for superior students; the other, the philosophy of modern psychology which recognizes the unsuitability of much of this material for less gifted students and would provide markedly different subject matter for them.

THE MATHEMATICAL COLLECTION

By GEORG WOLFF
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Introduction

Deductive reasoning versus inductive reasoning. Raphael's famous fresco "The School of Athens," which is in the so-called *Stanze* of the Vatican, illustrates the opposition between two schools of thought which have existed in all branches of learning since the ancients, and especially in philosophy and mathematics—deductive reasoning symbolized by Plato as opposed to inductive reasoning symbolized by Aristotle. Two of Raphael's pictures represent, on the one hand, the very strictest reign of logic in the science of numbers—in arithmetic, algebra, and analysis—and, on the other hand, the greater tolerance of geometry as an applied science. Euclid's compendium of the mathematics of his time followed Plato's steps, thus leaving little strictness of construction to be desired.

For centuries this scientific method of Euclid's has been used in schools as a method of teaching mathematics to young people, without any consideration of the fact that reasoning ability must first be developed and that according to psychology the way leads from the inductive process to the deductive.

Pedagogical science only has been interested in the important trend from primary emphasis on deduction to primary emphasis on induction. Johann Heinrich Pestalozzi (1746-1827), Johann Friedrich Herbart (1776-1841), and Adolf Diesterweg (1790-1866) came forward with their principles of *Anschauung* (intuition), and happened to be interested in pedagogy as well as in mathematics. If they had not been, who knows how mathematical teaching might have developed! Later the great pedagogue George Kerschensteiner (1854-1932) with his profound ideas on activity-instruction joined them as a friend and promoter of our subject.

The movement toward emphasis on intuition and activity. This movement to emphasize intuition and activity was chiefly a

pedagogical one, and was pushed forward by resolute didacticians of mathematical teaching, above all by Felix Klein and Peter Treutlein. Their names are closely connected with the well-known *Internationale Mathematische Unterrichts-Kommission* (I M U K).[1]

The necessity of changing the point of view of teachers in our subject has been thought very important. This is evident from the fact that at the fourth meeting at Cambridge, England, in 1912, a member of the central committee of the I M U K, Professor David Eugene Smith of Teachers College, Columbia University, gave an important report on "Intuition and Experiment in Mathematical Teaching in the Secondary Schools." The essential point of this report was the discussion of the question of how far strict system in geometry and algebra had been given up in the different countries of our globe. Professor David Eugene Smith stated that in Germany a considerable fusion between deductive and inductive reasoning could be observed and that in object lessons different means of instruction were being used for the purpose of making clear and deepening the abstract subject.

Fusion of deduction and induction. Since 1912 we in Germany have pursued this method of fusion, since it is the only one that arouses the interest and activity of youth. The heart of the matter is for us to go back to the inductive method; then to try to come to analysis. We do not proceed to deductive reasoning before a subject has been sufficiently developed in an inductive manner and before the young intellec is capable of receiving systematic statements. This method of instruction may be called the "physical method" on account of its close relation to the teaching of science. In consequence of this parallel to the sciences, we speak of demonstrational teaching, practical work, classroom work, laboratory work, and "mathematical collection." The substance of the new method is the collection of models, for by this means occasional practical work in the mathematical laboratory is possible. However, we do not think of transforming the whole teaching of mathematics into practical exercises—as is the case with physics, chemistry, and biology—but teaching by demonstration is at present of greatest importance to the pupils, and in this method models are certainly of great assistance.

The Englishman, John Perry (1850–1920), planned practical work in mathematics lessons, but there is a great difference between

[1] International Commission on the Teaching of Mathematics.

his and the modern point of view. Perry walked with seven-league boots across the matter of instruction. He attached little value to systematic advancement by the pupils from proposition to proposition, from rule to rule; he only informed them of the results, as in the theory of powers and roots, and introduced them to the technical application of these rules. This method may be practical for the training of reasoning power, but it cannot result in such good qualities in youth as those we intend to develop.

Educational System and Curriculum

The curriculum of the Herschelschule. A short account will be given of the organization of the mathematical collection at the Herschelschule,[2] Hannover. Before doing so, the outlines of the educational system of this school and of its curriculum for the teaching of mathematics must be traced.

The Herschelschule is an *Oberrealschule*. It has nine grades divided into three sections: lower course (first to third year), intermediate course (fourth to sixth year), and upper course (seventh to ninth year). Pupils enter the school at the average age of ten; they usually pass the final examination (the *Abitur*) at the age of nineteen. The following is the teaching curriculum:

LOWER COURSE

1. *Arithmetic.* Operations with whole numbers and fractions (common and decimal), reckoning in a concise manner, rates of interest.

2. *Geometry.* Beginnings of geometry, congruence, symmetry, the quadrangle up to the trapezium.

INTERMEDIATE COURSE

1. *Algebra.* Use of letters, theory of equations up to quadratics in one unknown quantity, proportions as equations, powers, roots, logarithms.

2. *Geometry.* Exercises on triangles and quadrangles, equivalent figures, the whole theory of the circle, similar figures, simple exercises on orthogonal projections, axonometry.

3. *Trigonometry.* Right-angled triangle, law of sine and cosine, practical exercises.

[2] The school is named after the great astronomer, Frederick William Herschel (1738-1822), born at Hannover.

UPPER COURSE

1. *Arithmetic.* Arithmetical progressions of first and higher order, finite and infinite geometric progressions, compound interest and annuities, statistics.

2. *Algebra.* Quadratic equations in two unknowns, solution of a cubic and other equations by graphs, table of values and nomography, elements of differential calculus up to the discussion of graphs, elements of integral calculus.

3. *Geometry.* Analytical geometry of the straight line, the circle, and the conics, producing of conics by cuts and by affine, collinear and commonly projective relationship, perspective, photogrammetry, difficult exercises on orthogonal projection.

4. *Trigonometry.* Difficult calculations in plane trigonometry field work, spherical trigonometry of the right-angled triangle, law of sine and cosine, astronomy with practical exercises.

This curriculum we call *Kernlehrplan* (curriculum essentials). With exceedingly clever classes these subjects may be increased, as there is also a *Randlehrplan* (bordering curriculum) that includes items such as analytic and geometric discussions of functions with complex variables, nomography, amplified statistics, and difficult exercises on integral calculus.

THE PARABOLA

General use of mathematical figures. The parabola will be discussed first in order to give an idea of the models, the means of instruction used with such a collection, and the mathematics lessons as a whole. It is very important to show the student mathematical figures, not only in their abstract shapes, but also in the great variety of forms and applications to be seen in daily life both indoors and outdoors, and in the apparatus of factories and laboratories: in short, mathematical forms evident in the surroundings of man wherever he may be.

Before explaining our ideas on the use of models during lessons, the subject of teaching must be considered for a moment. When a pupil, the writer learned only the old Greek definition of a parabola which is founded upon the focus and directrix (Fig. 4). He became acquainted with many nice qualities of this curve, open on one side, but did not obtain a complete idea of the parabola until after leaving the secondary school. The present German

generation is better off. Another example of the new method is used in the intermediate course when pupils learn the power function $y = x^2$ (Fig. 1) and to understand variables like $y = -x^2$. They also understand $x^2 = y$ and $x^2 = -y$ without difficulty.

Generation of the parabola. At the *Oberrealschule* we try to make pupils understand the methods of generation which modern geometry gave us more than a hundred years ago. They have the advantage of variety and clarity. The generation of the parabola by a projective range of points (Fig. 5) and the special cases resulting from it, like Fig. 3, must be mentioned here.

The generation of a parabola by envelopes is another related method; these envelopes are tangents (Fig. 2) and are, therefore,

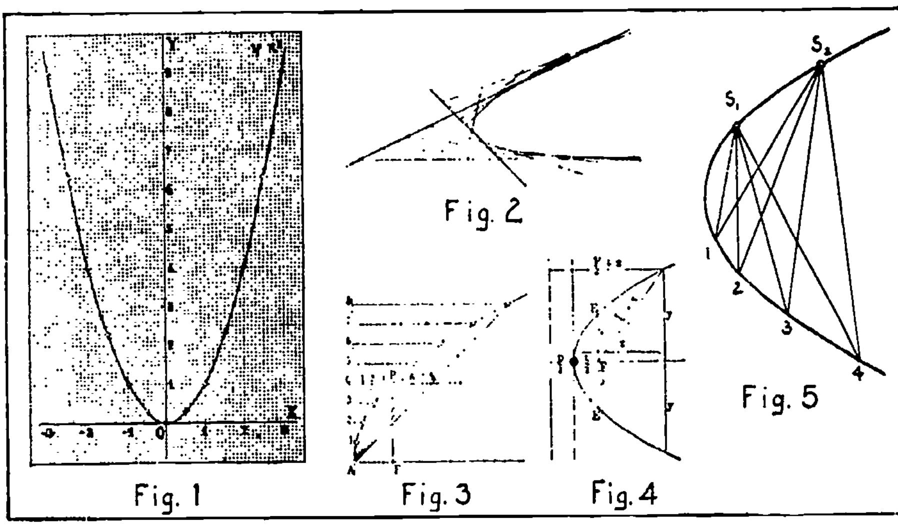


Fig. 1 Fig. 2 Fig. 3 Fig. 4 Fig. 5

the dual case of that shown in Fig. 4, the generation by a projective range of points.

But there is still another method. The question, keeping in mind the idea of the transformation of figures in Poncelet's sense, is how to get a parabola by affine projection. It is only possible by another parabola.

If these different methods of generation are carried through during the class period and if the pupils actually draw the curve according to these directions with their own hands, they will surely understand the parabola better than we did in our school days.

Examples of the use of models to enlarge geometric conceptions. The models are found to be of still further assistance in the enlargement of geometric conceptions. Those which we see

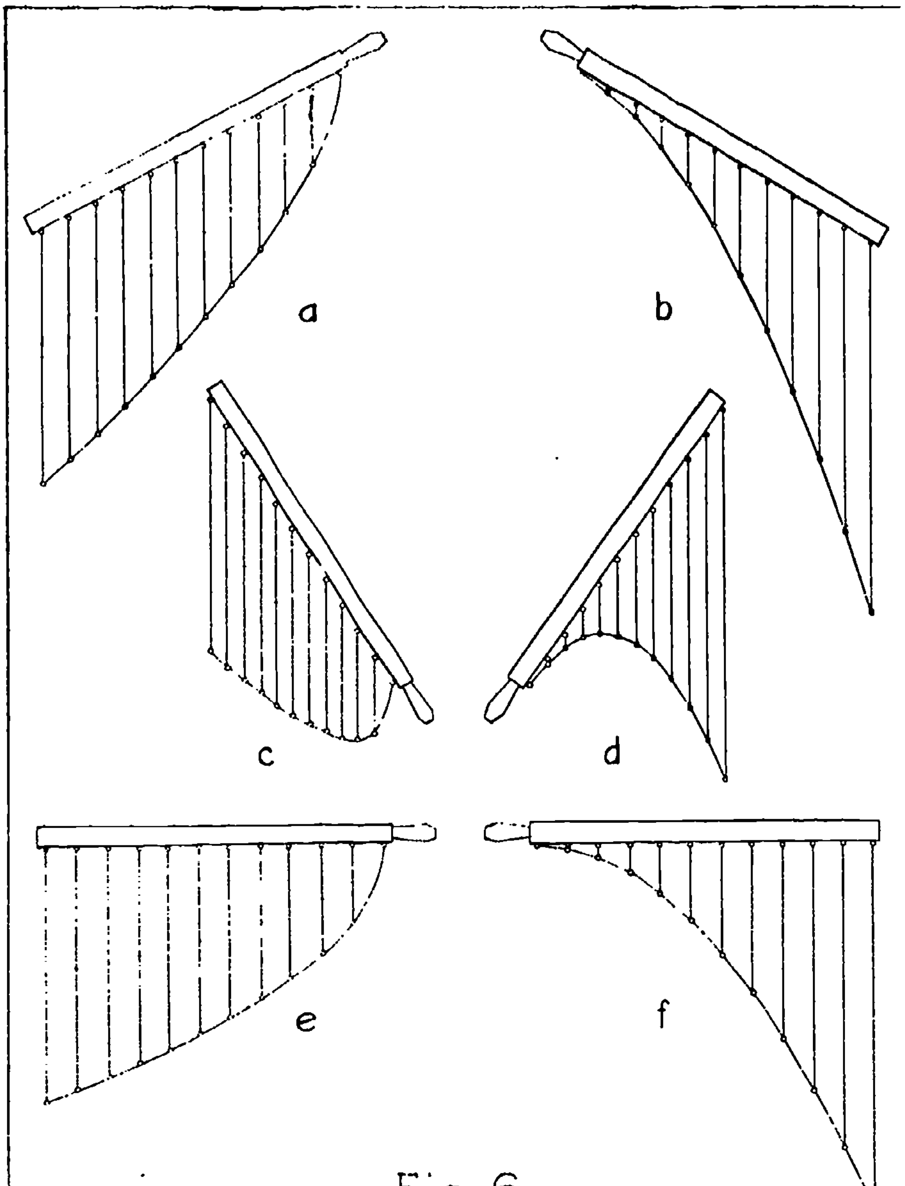


Fig. 6

in Figs. 6e and 6f are especially useful if we turn the bars to which the rods are fastened. The Figs. 6a, 6b, 6c, and 6d illustrate the operation. If we imagine the initial motion continued in each case, we finally see the parabola become a straight line.

Nor is a use of such models sufficient. Experiments, other models, and a study of the relationship of geometric forms to the

external world all lead to a deeper insight into the subject. The work is planned in this way to necessitate deduction and to provide examples of application, which cannot be discussed here, before theoretical treatment. The exact arrangement is a matter of taste,

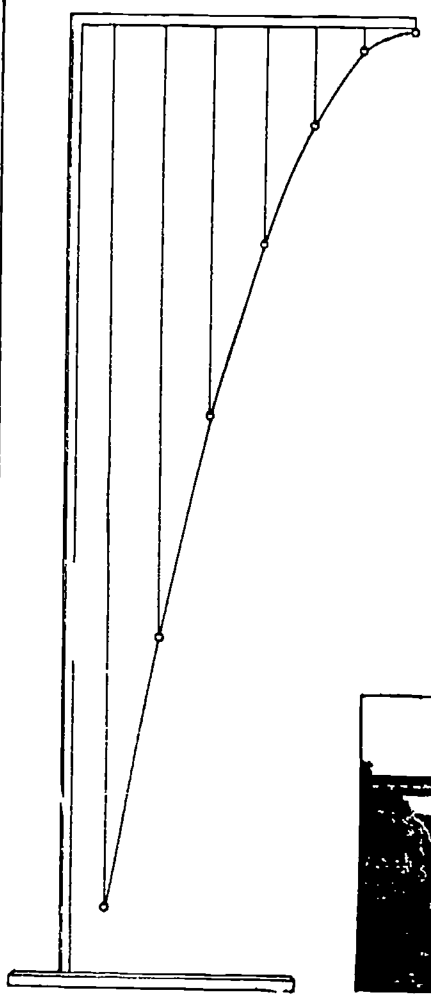


Fig. 7

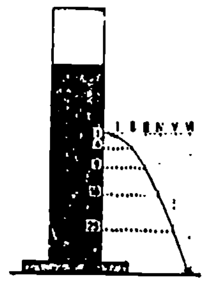


Fig. 8

Fig. 9

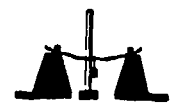


Fig. 10

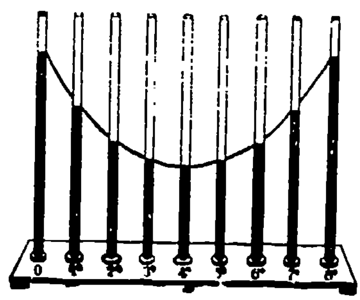


Fig. 11

Fig. 12

of disposition, of personal point of view. One cannot proceed in the same manner every year.

Every pupil has observed the stream of water that makes a "bend" as it comes from a hose. This curve is shown by an experiment (Fig. 8). Bolts and projectiles (Fig. 9) fly in a ballistic curve which is nearly a parabola. When metal is deflected (Fig. 10),

a parabola can appear. Finally, we study something connected with the subject of falling bodies, namely, that curve which results when the spaces traversed in equal portions of time are represented by hanging rods (Fig. 7).

Examining the change of volume of water between 0° and 8 centigrade when discussing the theory of heat, we find the parabola indicated in Fig. 11.

When we were studying the conic sections in this way, a pupil produced Fig. 12, and verified the fact that there was a parabola by pointing out the vertical bars involved.

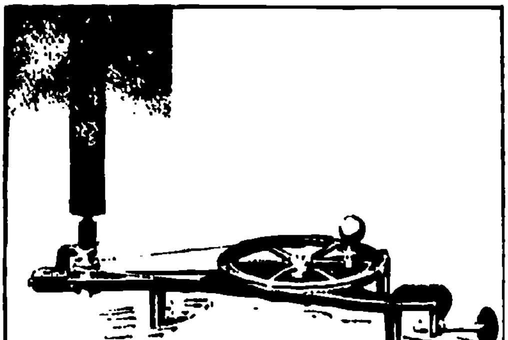


Fig. 13

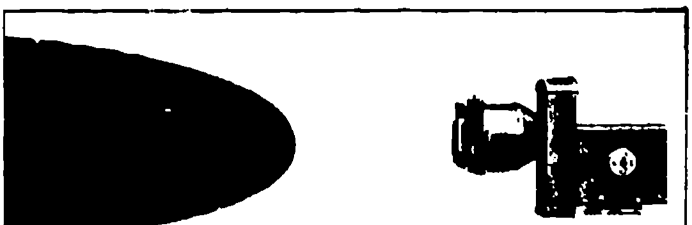


Fig. 14

When a cylinder containing some liquid is turned on a whirling-machine (Fig. 13), a parabola is distinctly to be seen. This gives the first occasion for the mention of that solid figure which is the result in reality, the paraboloid. We use the same whirling-machine and put on it a parabola cut out of metal.[3] By rotation we obtain the parabolic solid (Fig. 15) of the equation $x^2 + y^2 - 2pz = 0$, its projection (Fig. 16) being a circle (see also Fig. 17). When we discussed this generation of a paraboloid, a pupil presented the two

[3] All the above-mentioned models can be obtained from the Schul-Verlagshandlung Kreye, Fernroderstr. 16, Hannover, Germany.

plaster models of paraboloids (Figs. 18 and 19) depending upon the parabolas $y^2 = 2z$ and $y^2 = 4z$.

So the problem of the elliptic paraboloid suggested itself. Its equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b} - 2pz = 0.$$

After this digression into solid geometry, we return to other methods of generating a parabola. The fact that the sections of the above-mentioned paraboloids produced parabolas suggested the question of how to obtain this curve by other sections. The experiment with a conical glass (Fig. 23) filled with a dark liquid shows the parabolic form. Germinal Pierre Dandelin (1794-1847), as is known, inscribed a sphere in a cone (Fig. 20) with the aid of which one can obtain the analytic form of a parabola without extraordinary difficulty. If we intend to connect the parabolic section with central collineation, the model (Fig. 21) will be of great use. The movement (Fig. 22) shows the perspective relationship which exists between the circular base and the section with the vertex of the cone as center.

The crown of geometric research still remains the central collinear drawing which we used at the last. The sketch in Fig. 14 shows in the form of a diagram how to get the figure of a parabola with the aid of a projector and a circular disk of metal or wood. The brush of rays and the position of the circle is demonstrated very distinctly in Fig. 24.

General purpose of the method. Let us cast a backward glance at this brief account. The essential point is *observation of functions in the widest sense.* On the one hand, the various forms of the parabola in itself are to be demonstrated quite clearly to the pupil, as the many forms shown to him indicate. On the other hand, the pupil has to realize fully the diversity of generation. Therefore, the curve itself and the methods of generation are seen to be actually functional.

We must cease to show the pupil the mathematical figures in the textbook or on the blackboard. He must draw them himself; he must take the models in his hand and study with his own hand and eye what takes place. The experiments described must either be executed by him or demonstrated to him in the mathematics lessons. Then his eyes, and his heart too, will be opened to the curve; then he will get a vision of space in the world around him.


Fig. 15

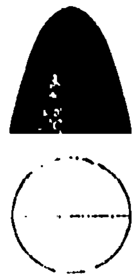


Fig. 16


Fig. 17


Fig. 18


Fig. 19


Fig. 20

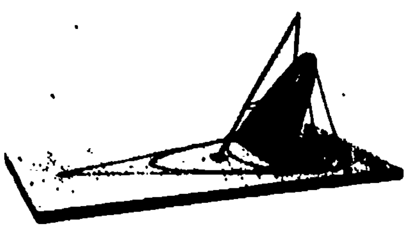


Fig. 21

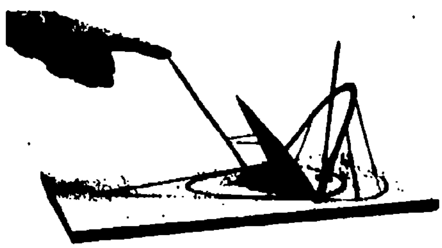


Fig. 22

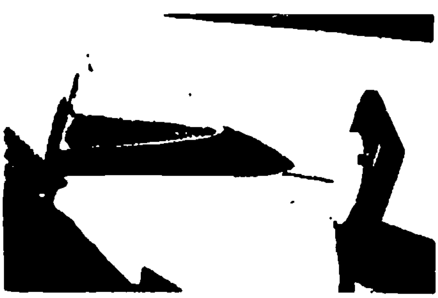


Fig. 23

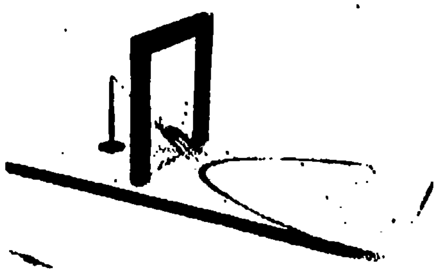


Fig. 24

General Remarks on the Use of Models

Usefulness of models in mathematical instruction. A further discussion of the models referred to above is necessary to explain their use in mathematical instruction. This may be demonstrated in

a general as well as a special sense. Generally speaking, there is great educational value in the methodical and pedagogic aspects of the use of models: specifically, we have to acknowledge the didactic and logical assistance of such material.

Models may be used as follows:

1. Mathematical qualities can be symbolized by real actions. Here we mention Figs. 6 and 23.

2. Understanding of difficult processes of reasoning can be facilitated by empiric help. This purpose is served by Figs. 14 to 22, and 24.

3. The possibilities of applying mathematics can be shown in the pure and applied sciences. The models shown in Fig. 13 and in Figs. 7 to 12 are suited to this purpose.

4. The usefulness of mathematics in social life can be proved. Here we mention the many games that use small sticks and plates, as well as the arithmetical and geometric games.

Models can be employed in teaching in the following ways:

1. To prepare a deduction. Example: If a paraboloid has been generated by rotation (Fig. 15), its equation can be easily deduced.

2. For the deduction itself. The typical example is the model of Dandelin's sphere (Fig. 20). The qualities of a parabola are scarcely to be comprehended without this means of instruction. The experiment illustrated in Fig. 8 is a good example.

3. To prove a deduction. If we have demonstrated, for example, the proposition of the similarity of all parabolas, the models in Figs. 6*a*, 6*c*, and 6*e*, with the indicated points, will show this quality.

4. To prove theoretical reasoning. The relationship between circle and parabola is to be deduced in quite an abstract manner. The models in Figs. 21, 22, and 24 illustrate the truth of this reasoning. The apparatus in Figs. 7 to 10 can be employed likewise.

5. To widen the subject dealt with. In the first place examine Fig. 12. The right use of models relieves mathematics of its intrinsic rigidity. It clarifies ideas and notions according to the famous words of our Königsberg philosopher Emmanuel Kant: *Begriffe ohne Anschauung sind leer,* that is, "Ideas without views (perceptions) are empty."

Arithmetic and Algebra

Further examples of the use of models. Some models from the different branches of school mathematics will now be described

according to their purpose and use. It is not possible to name or present all the apparatus which we have collected in the course of time. Such an enumeration would be too long and of no general interest. The intention here is, in the main, to indicate some novelty.

In arithmetical teaching, the abacus in the form of a soroban can be used according to the example of the Asiatic nations. In Figs. 25*a* and 25*b* we see a ledge dividing the case into two parts. In Fig. 25*a* each ball above the ledge means ten units: in conse-

a Fig. 25 b

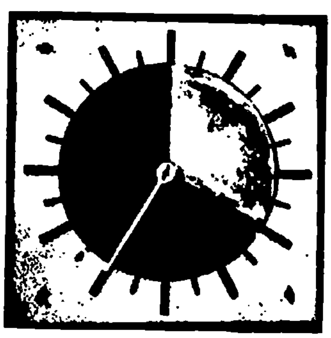


Fig. 26

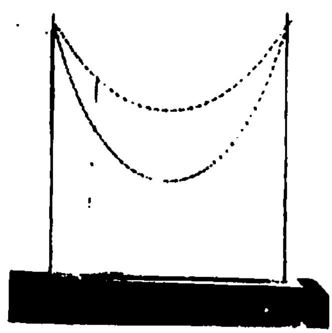


Fig. 27

quence, in the lower part of the case ten balls are to be found on each rod. The balls pushed upward are counted and the number 48 can be read off on this model.

On the second abacus five balls are below and two are above. Each upper ball means five units. The number 68 is easily read off. From this abacus the mechanical calculator has developed.

Fig. 28 shows our idea of how to demonstrate and make clear the simplest matters of instruction. These scales are taken into the classroom, and water or something else is weighed, so that the pupils learn to use weights.

The nomogram (Fig. 30) can do good service in the addition and subtraction of whole numbers and fractions.

Dr. Spitta has constructed the apparatus of Fig. 26 after an old model designed to demonstrate fractions. We see that the rim is divided into twenty-four equal parts. At the present moment the fractions 1/3 and 2/3, distinctly set off by colors, may be read. The apparatus is adjusted by turning the white lever.

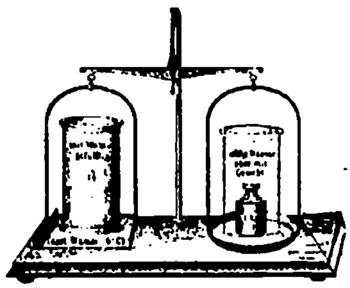


Fig. 28

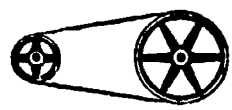


Fig. 29

Proportion forms an important subject. Though we have reduced proportions entirely to linear equations, the difficulty of obtaining a clear understanding of ratio cannot be denied. Here we

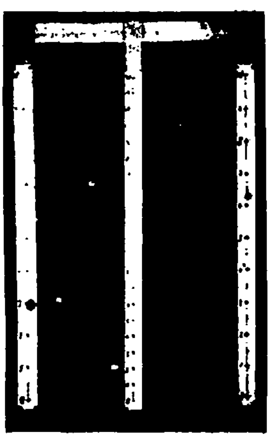


Fig. 30

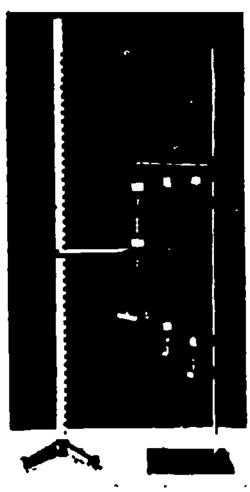


Fig. 31

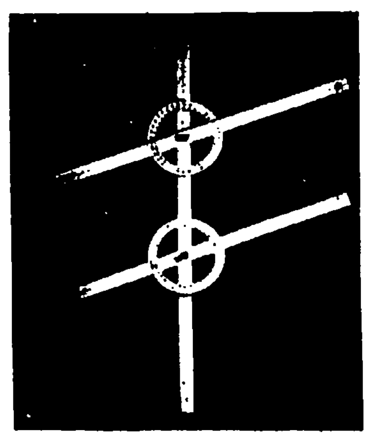


Fig. 32

see a very good example of practical exercises done by pupils. Fig. 31 shows an arrangement of which we do not think a precise description and explanation necessary, since extension is known to be in proportion to weight. The proportional factor is easily found.

The model shown in Fig. 29 may be of use to demonstrate in-

verse proportionality; the smaller the radius, the greater is the number of revolutions, and vice versa.

That proportional compasses must be used in a study of the theory of proportion goes without saying. They are often reproduced in American textbooks.

The theory of proportion affords a very good example of practical use according to the methodical principle of "concentration,"[4] as proportions were used by artists for drawings of men and women in the days of the Renaissance when man was regarded as the measure of all, and when the chief aim of artists was to model the human body in an ideal form. We know that Leonardo da Vinci (1452-1519) diligently studied proportion in order to create beautiful bodies. These proportions of the human body were analyzed in the time of the ancients. Great tables were drawn up showing the relations of all parts to one another and especially of all parts to the whole body. Such a table was called a "canon."

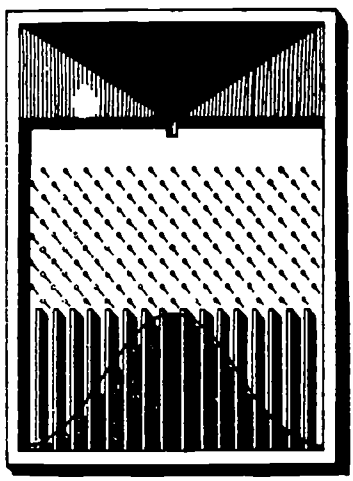


Fig. 33

One sketch by Leonardo represents a head. In it one can see that the part above the forehead is equal to the breadth of eyes and to the distance from nose to mouth. Likewise we find the same distance from eye to point of nose and from mouth to chin. Moreover, we see that the shorter distance is contained twice in the space from mouth to chin and three times in the forehead. From a similar sketch, of a leg, we can also read off some proportions.

But we must not think that every painter proceeded in the same way so that every body was similarly constructed; with other painters as with Leonardo we find several kinds of measuring tables. The German painter Albrecht Dürer (1471-1528) also designed his human bodies by *Zirkel und Richtscheidt*, i.e., by compasses and ruler.

For the teaching of algebra we have additional models: a slide

[4] See *Richtlinien für die Lehrpläne der höheren Schulen Preussens (Course of Study for the Secondary Schools of Prussia)*. Weidmannsche Buchhandlung, Berlin, 1931.

rule, two catenaries (Fig. 27)– the parallel rods can be adjusted narrower and wider—and finally Galton's board (Fig. 33) for the calculus of probabilities. At this board practical exercises can be undertaken in an experimental way to study the curve of the rule of partnership as stated by Gauss, leading later to a theoretical consideration. The equation reads:

$$y = y_0 \cdot l^{-\frac{x^2}{2v^2}}$$

GEOMETRY FOR THE INTERMEDIATE COURSE

Models used in plane geometry. As a first example let us describe that model by which angles of parallels can be studied (Fig. 32). There are two parallel straight lines and another one bisecting them. The problem is to compare the magnitude of the original angles.

To read off the angles, we have set down the points of intersection as centers of full circles. For practical purposes the model nas been constructed of slats which can be pivoted. The parallel slats are connected with wires, and on these are drawn the three lines mentioned.

The different angles are read off in various positions of the straight line and the parallels, their magnitudes are compared, and so we find out the well-known relations between alternate and other angles. This can be done either by obvious demonstration or as a practical exercise by pupils. We think an abstract proof is not necessary. The empiric comprehension shows the mathematical quality quite distinctly.

Models used in solid geometry. Now for the solids! We attach much value to solids of different sizes and of different materials, right and oblique. The drawings (Figs. 34*a* and 34*b*) show some of these solids. We teachers know that in solid geometry the great difficulty is the transition from prisms to pyramids. The principle of Francesco Bonaventura Cavalieri 1598?-1647) helps over this difficulty (Fig. 35). The essential point is that here careful reasoning about transition (*Grenzbetrachtung*) is used, without which we cannot succeed. Formerly the pyramids and prisms were only constructed of small plates 1 cm. high. Nowadays we begin with these models too, it is true, but we have these small plates made of the thinnest paper in order that we may omit the well-known steps at the lateral edges of the pyramid. Here

it becomes evident how profoundly the pupil must think over the process of transition, so that he will understand it clearly.

The result of the indicated reflection is the theorem that the

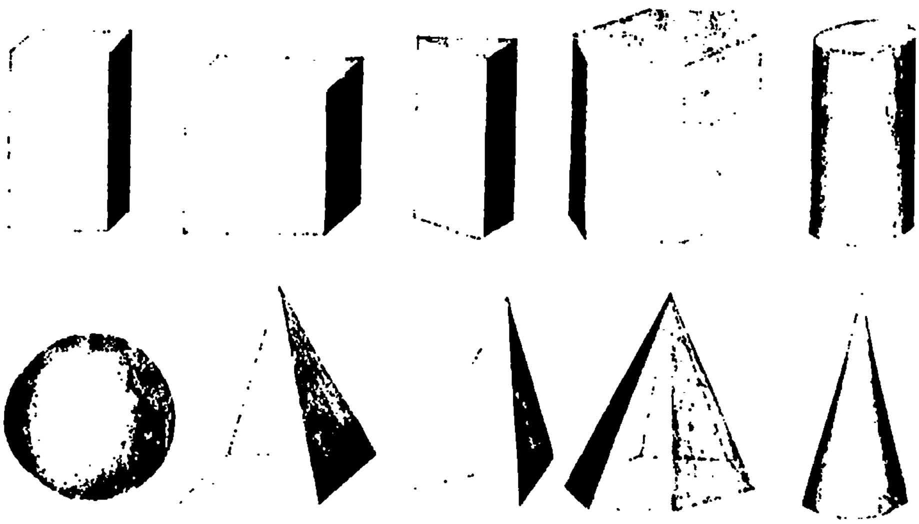


Fig 34a

Fig 34b

volume of a prism is three times as great as that of a pyramid of equal base and equal height.

For this theorem we have very valuable tin models, with which the experimental proof by sand or water is done in the classroom or as a practical exercise by the pupils (Figs. 36 and 37). The

solids belonging together can easily be discovered. Fig. 37 shows how the prism is supplied with sand up to the edge by filling the pyramid three times.

The most important step for pupils working with solids is the

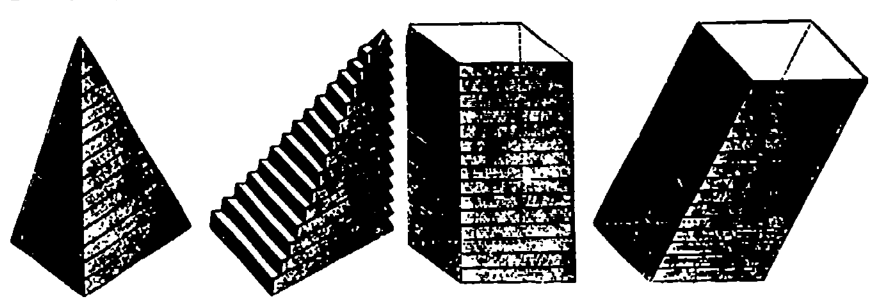


Fig. 35

construction of these solids with their own hands, using small sticks. For this purpose we have special cases filled with small sticks and "plastilina."[5] The sample on the case (Fig. 39) shows

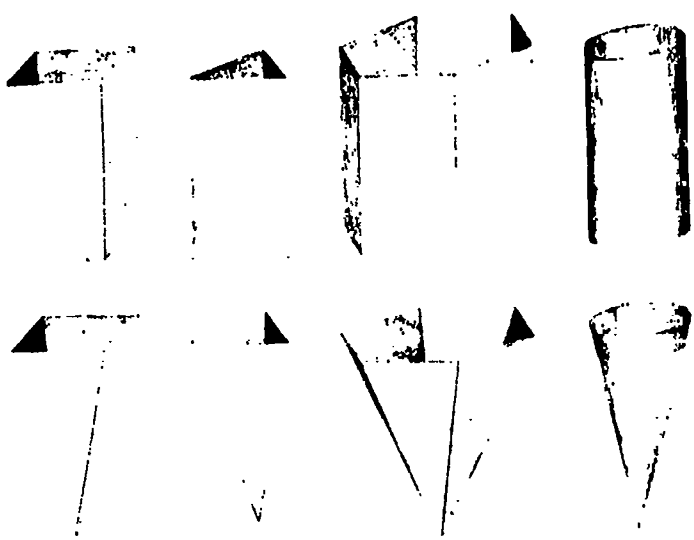


Fig. 36

the right use of the materials within. Fig. 38 shows other examples of such solids.

It is surprising how eagerly the pupils work with these small sticks (at Hannover called *Wurstspieler*). Activity begins when

[5] Modeling clay.

pupils construct with their own hands, come into contact with the solid, and see the possibility of putting in some colored sticks representing diagonals and heights of planes or of solids. With the aid of his model the pupil is able to ascertain the triangle of a given height, i.e., slant height and to calculate the missing part. This personal activity stimulates coöperation extraordinarily, and we consider rousing the pupil's interest an important part of our teaching.

Fig 37

The principle of universal functionality also deserves notice in connection with solids. The sphere, the cylinder, and the cone (Fig. 40) have to be generated by the revolution of a circle, a rectangle, and an isosceles triangle on the whirling-machine.

By means of sand and funnels (Fig. 41) we form cones and

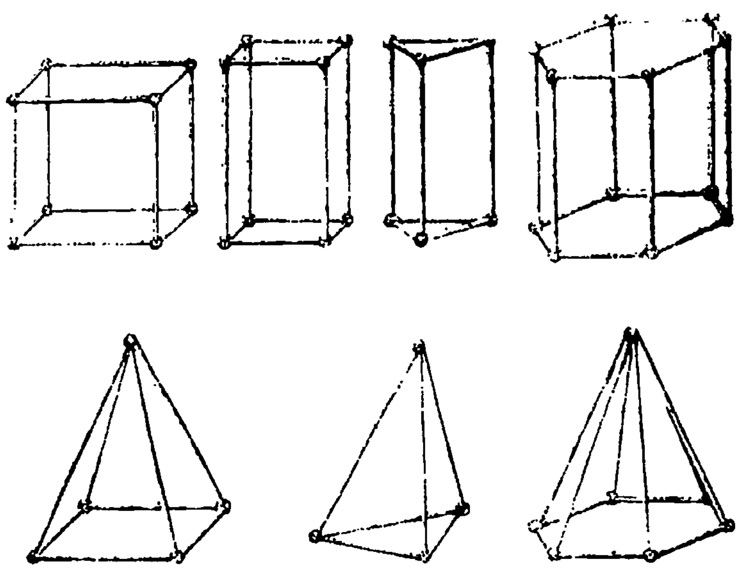


Fig 38

their osculant line as a curve of three dimensions and of the fourth order.

Finally, we use the thread model of a cylinder to show the pupils

the ever-amazing transition from the cylinder to the one-shelled hyperboloid and to the double cone (Figs. 42*a*, 42*b*, and 42*c*).

To be quite clear, something should be added to this account of solid geometry. With us in Germany the principle of teaching geometry in the sense of dealing with solids prevails more and more. We not only study angles, lines, distances, and figures in two dimensions, but we also lay great stress upon continually changing planes and solid bodies. After deducing a proposition about

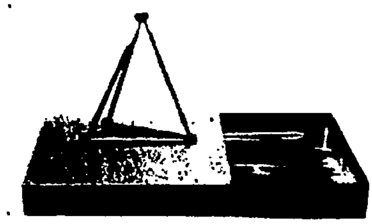


Fig. 39

Fig. 40

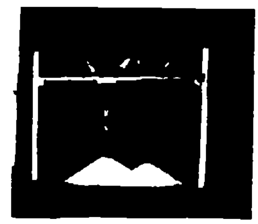
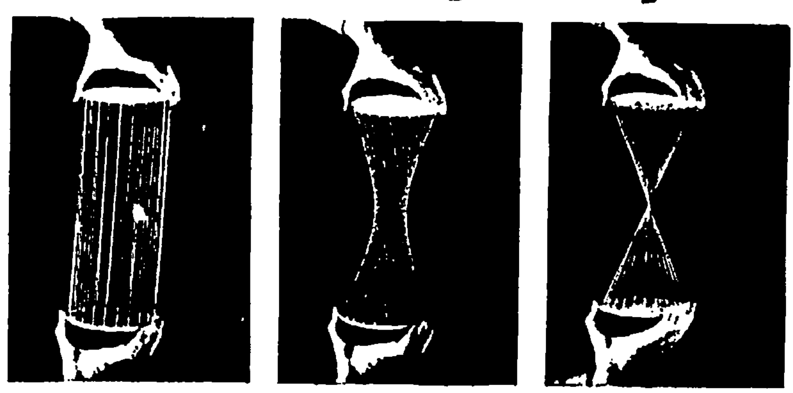


Fig. 41



a b c

Fig. 42

the isosceles triangle, we search all solid bodies for this triangle in order to get a clear picture of what we have learned. Following Schimmack, we call this method of teaching geometry "fusion," that is to say, a continual connection between solid and plane reasoning.

These words of explanation were thought necessary before going on, for precise mathematical particulars cannot be given without a discussion of many more models, and because of the confined space available here. The only additional comment to be made will be that the above-mentioned method of using sticks is applied to plane geometry as well as to solid.

Additional discussion of models used in plane geometry. For plane geometry still other apparatus must be used for the purposes of illustration and explanation. The relations of quadrangles only will be discussed.

In several German towns the cyclists use direction pointers so that they need not lift their hands from the handlebars at crossings. This apparatus is represented in Fig. 43. The upper figure shows the rhombus-shaped links; at the right is a white and red disk, and at the left is the handle for adjusting. In the lower figure the disk is projected. This is the position from which to start when teaching. The links form squares. In drawing back the disk, the squares change into equilateral quadrangles of various forms.

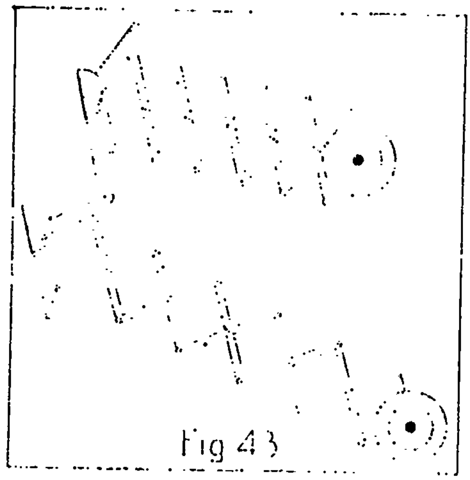


Fig 43

This transition is used in Fig. 44 to introduce a diagonal and to study the angles. The next model contains two diagonals.

For the illustration of the relationships of angles and diagonals we have constructed a model (Fig. 45) which is adjustable. We

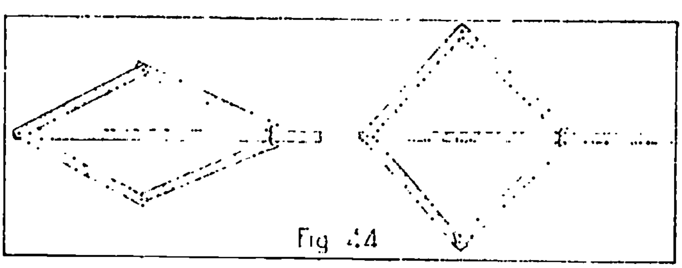


Fig 44

can make the sides longer and shorter, and we can shift the sides correspondingly; at the same time we have two diagonals. The sketches 45*a* to 45*f* illustrate this clearly. We see *a* the rectangle, *b* the square, *c* the parallelogram, *d* the rhombus, *e* the trapezium, and *f* the quadrilateral.

Of course there are models for the theory of circles. Some sketches are shown. They concern the theorem of Thales which deals with the right angle in the semicircle (Fig. 46*a*), the re-

lation between the inscribed angle and the angle between the chord and the tangent (Fig. 46*b*), and the angle at the center (Fig. 46*c*).

When dealing with the sphere, difficulty arises in finding out the number π. According to modern ideas in mathematical instruction, the teacher's main duty is to present the idea of limit. Here

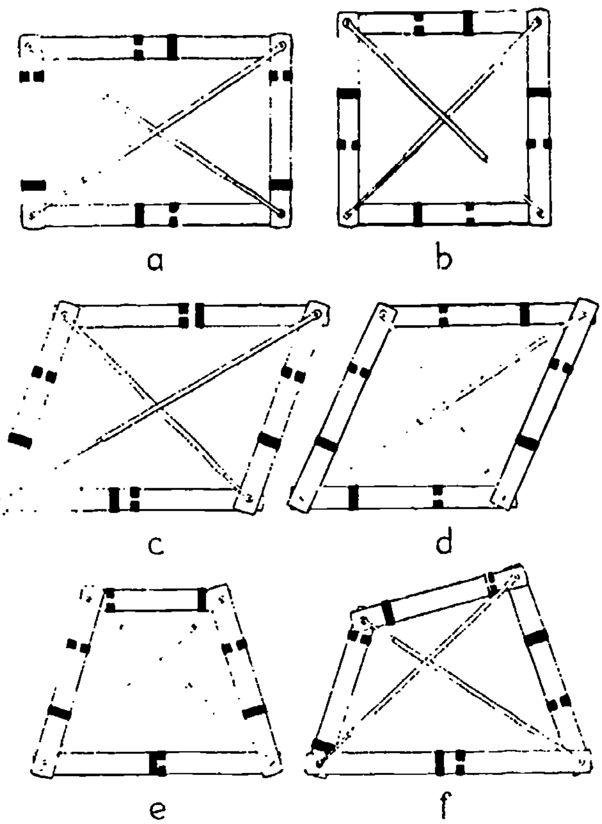


Fig. 45

a practical exercise is of great importance. We have wooden disks of various radii. The pupils measure the circumference (c) with a tailor's tape measure and the radius or the diameter ($2r$); then they try to find out the value of $c/2r$ on different disks. The result of course can be only approximate; the point of this exercise is for the pupil to realize the constancy of this quotient, however large or small the disk may be. In order to give the pupils an idea of the transcendental and irrational character of the number π,

we have indicated the value of π up to fifty decimal places in the form of a frieze on the wall of our collection room.

At the conclusion of this account of the intermediate course

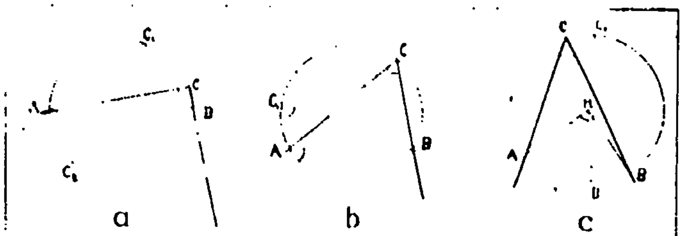


Fig 46

two pieces of apparatus should be described. In Fig. 47 we see a model for the changing of surfaces. A and B are fixed points, AD and BC are elastic. DC slides in a wooden slat of the size of AB. To the small slat in which DC slides hooks are fastened which can be shifted on a longer bar. By shifting, the rectangle $ABCD$ was made into the parallelogram $ABEF$ of equal area. When dealing with geometric proportions we show and use the diagonal scale. On our maps we have different scales, e.g. 1 :1000, 1 :625, 1 :2000, and so forth. Practical exercises are done with them.

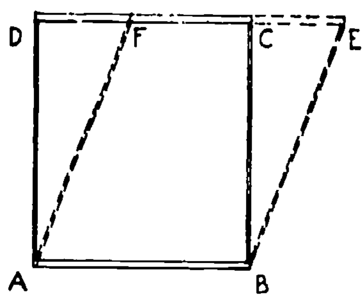


Fig. 47

Besides the models mentioned already, there are of course several others such as pantographs, but they are to be found in every mathematical collection.

Geometry for the Upper Course

Unification of the subject of geometry. The great reform of mathematical teaching under the leadership of Felix Klein, David Eugene Smith, and others first prompted the teaching of analytics in the secondary schools. In geometry, the problem of introducing the subject and the problem of educating teachers to include the study of space was discussed, for the useful unification which arithmetic, algebra, and analytics had undergone had found no parallel on the side of geometry. The reform in the teaching of

geometry was followed by a movement to lay great stress on the promotion of the subject in secondary schools. A "silent reform" began, as we say. This movement tried in the first place to amalgamate the various branches of geometry: Euclidean, analytic, projective, descriptive, and Apollonian geometry.

The importance of projection in geometry. Felix Klein in his famous *Erlanger Programm*[6] has indicated the importance of the projective group in the unification of geometry. The entire subject of geometry may be divided into five topics: *congruence*, *symmetry*, *similarity*, *affinity*, and *collineation*. This sequence is divided in such a way that in the intermediate course the first three predominate, in the upper course the last two. The affine and central-collinear transformation as part of the projective aspect form the connection between the conic sections and the cylinder and the cone, in a word, between the parallel-projective and central-perspective ways of generating conics. And thus an important connecting link to descriptive geometry is created.

The affine relation between two prismatic sections is shown by the model in Fig. 48*a*. The connecting lines of corresponding points are parallel and the points of intersection of corresponding straight lines are situated on the so-called axis, the section-line of the two section-planes.

Now it is important that this relationship remains when we turn one section-plane. The hand holds this plane straight in turning. Having arrived at the level of the other section-plane, we see that the affinity at first defined for three dimensions has also validity for two. This plane affinity produces other relations.

Keeping in mind these relations, the models of Figs. 48*b*, 48*c*, 48*d*, 48*e*, and 49*c* are based. They indicate the affine relations between circle and ellipse, and between ellipse and ellipse.

The highest stage of affine projection is that of central collineation. Instead of putting water into a cylinder as in Fig. 48*e*, we have to use the cone (Fig. 48*f*) and obtain the ellipse. The collinear relation is obviously shown by the projective threads in Fig. 48*g*.

Perspective position and three dimensions. Either at this time or at another a clear understanding of the famous theorem of Girard Desargues (1593-1661) is of high importance. As is known, the problem is to find out the condition for the perspective position

[6] Klein, Felix, *Gesammelte Werke*, Vol. I, pp. 411 ff. and pp. 460 ff., Julius Springer, Berlin, 1921.

of two figures. The theorem has validity for planes and for solids. The model (Figs. 48*h* and 48*i*) makes this relationship clear. Probably this turning apparatus is the most important means of teaching objective lessons in our mathematical collection. It illustrates affinity, similarity, symmetry, and congruence, and shows clearly the relation in the second and third dimensions.

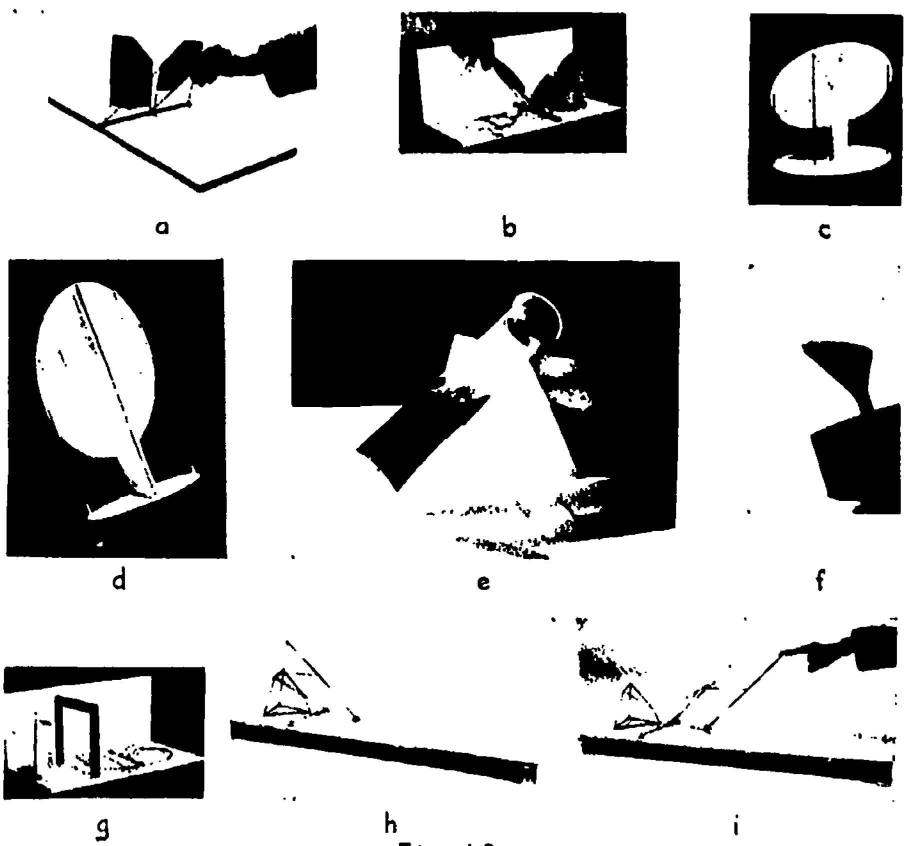


Fig. 48

The models shown in Figs. 49*a* and 49*b* indicate the pedagogical importance of the train of thoughts discussed above, for the understanding of three dimensions in the relationship between circle and hyperbola is extremely difficult. The projective lines in Fig. 49*a* clarify this excursion into the third dimension, which at other times is very troublesome; Fig. 49*b* has the same effect.

The models according to Dandelin (page 224) are represented in Figs. 49*c*, 49*d*, and 49*e*, as a sort of summary. Once more we see the projective and central-perspective relationship of three dimen-

sions. We have already become acquainted with the models for the study of the parabola (Figs. 14 to 24).

Miscellaneous illustrative models for the upper course. As to the ellipsoid and the one-shelled hyperboloid, we refer to their generation by means of the whirling-machine. Of course there are a considerable number of models of wood and gypsum for the various ellipsoids and hyperboloids in our collection room.

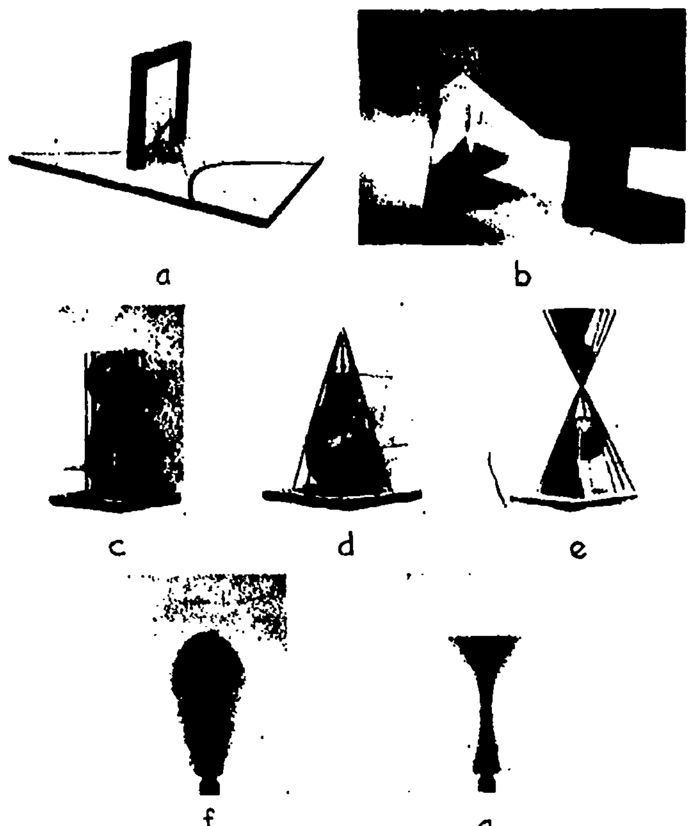


Fig. 49

The next six models are related to the parallel and to central projection in a special sense. In Fig. 50*a* can be seen a cone in different positions. In Figs. 50*b* and 50*c* the means of discovering the real length of a straight line from its projection is shown. The advantage of this arrangement is that the line may be turned down into the horizontal plane as well as kept in an upright projection. Since the supporting triangle is difficult to conceive, it is represented in Fig. 50*d*. With one finger the triangle is held

and with another a separate hypotenuse. On the horizontal plane the way of turning down the hypotenuse and with it the means of generation are indicated by which the gradient of slope, in other words the supporting triangle, may be obtained. The area of a triangle, as is known, is to be found from its projections. The model

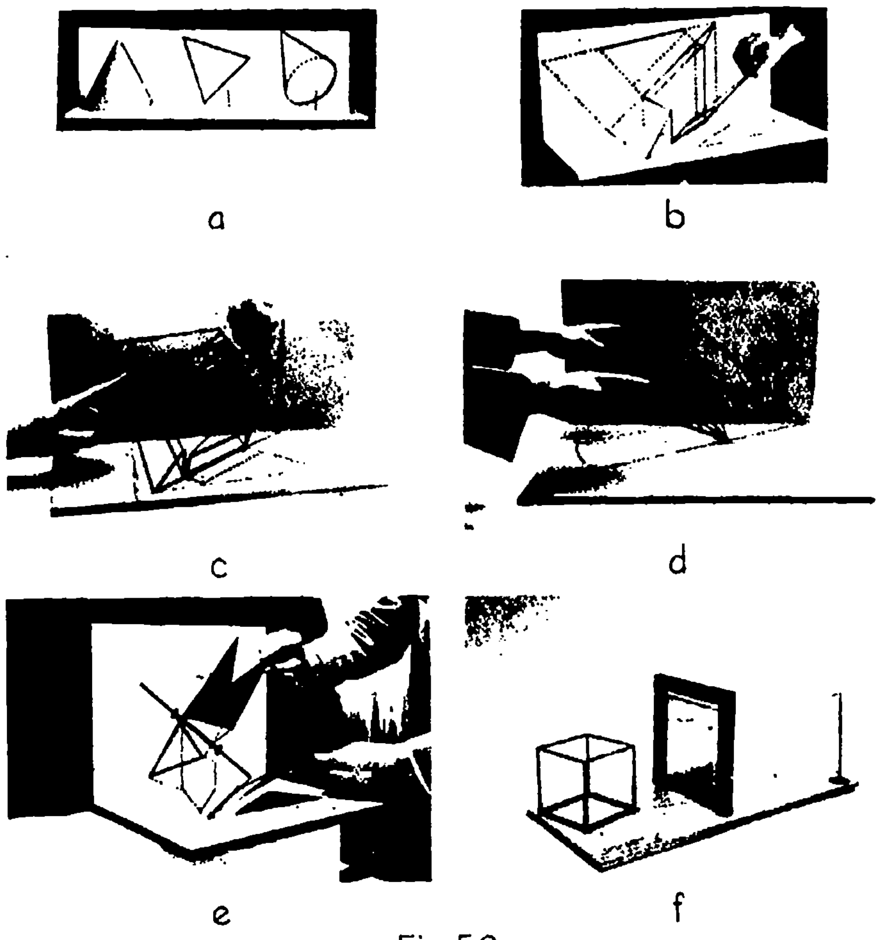


Fig. 50

in Fig. 50c may be of use in solving this problem. The demonstration of the perspective of a cube may be mentioned as the last of this group.

In spherical trigonometry we have necessarily to consider the celestial vault (Fig. 51) and its orientation. The development of ideas about the universe and the heavenly bodies is of historic interest. The difficulty which the so-called loop line of the planets caused all their interpreters leads us to Claudius Ptolemæus of

Alexandria (200 B.C.) and his theory of cycloids which we make use of now and then.

The Landheim. Our school is well off in possessing a *Landheim*,[7] where we are able to take up celestial studies and the application of mathematics to field work. The teaching of mathematics can be very successful in the *Landheim*. Apparatus for both

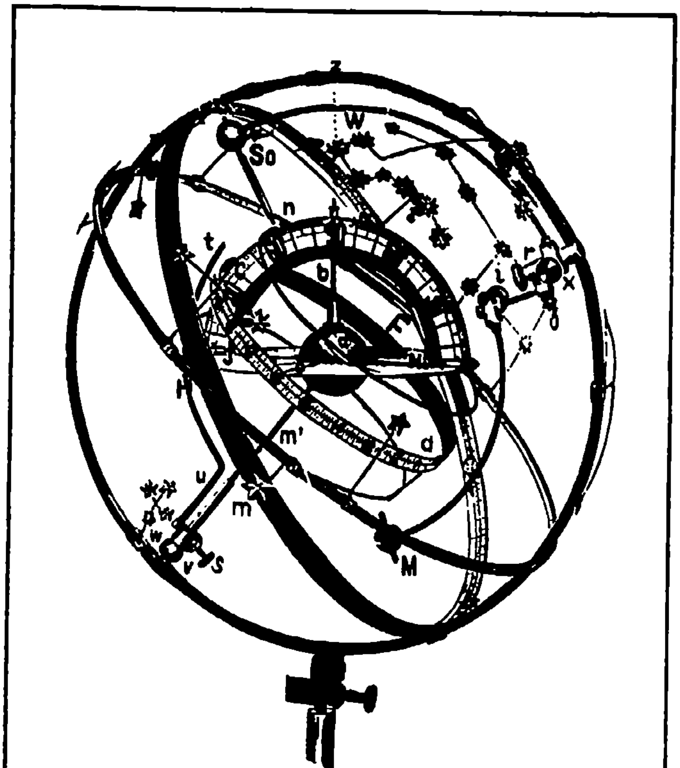


Fig. 51

simple and difficult measurement must be available there, such as tape measure, theodolite, surveyor's table, and leveling instruments. We strive to construct the measuring apparatus ourselves, if possible, in our workroom. In Fig. 52 the latest products of the Herschelschule in this respect are to be seen.

[7] A *Landheim* is a house for pupils in the country where they go for a fortnight every year during the school term. They are taught there as well as in town, but the principal matter of interest is that the instruction refers to the country and its inhabitants, to civilization, to the surface of the ground, and the like.

Conclusion

In the above account of our models only a selection of such things as appeared essential has been made. Besides the models, apparatus, sketches, and tables, we have a collection of lantern slides, containing pictures showing the relationships between art and per-

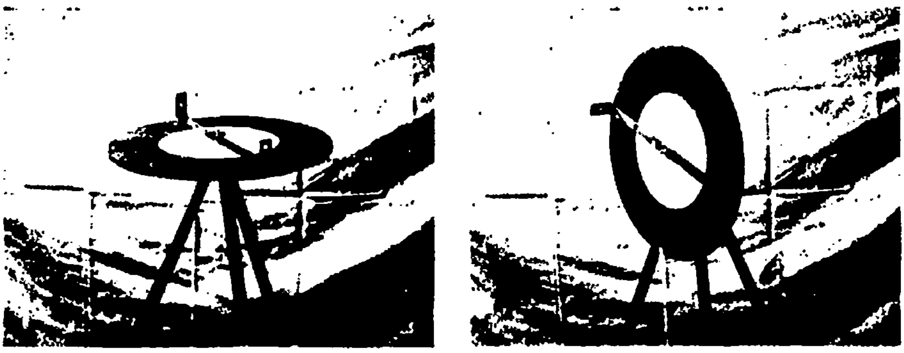


Fig. 52

spective, proportions in art, allegorical representations of mathematics, and finally a large number of portraits of great mathematicians.

All this we have brought together to make mathematics pleasing and concrete. The well-known discrete curve of Karl Th W.

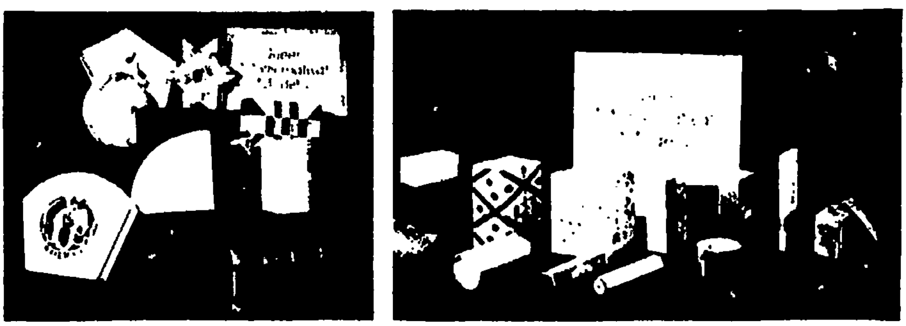


Fig. 53

Weyerstrass (1815-1897) obviously showed that abstract *Gehirnmathematik* (brain mathematics) is of no use in school. Pedagogics, psychology, and experience in teaching demand that we teach demonstrative mathematics, in the sense of educating the pupil to good mathematical reasoning. From empiric to abstract subjects—that is the way!

In our collection there is also an amusing assortment of models composed of chocolate boxes (Fig. 53). Sweet mathematics

UNITS OF INSTRUCTION IN SECONDARY MATHEMATICS

THE UNIT, DIRECT LINEAR VARIATION

By J. S. GEORGES
Crane Junior College, Chicago, Illinois

General principles of unitary organization. The organization of instructional materials into units has for its main purpose the enhancement of the value of mathematical instruction, and the facilitation of the learning of mathematical concepts, principles, and processes. It undertakes to group together homogeneous materials which are different aspects and manifestations of some basic mathematical principle or concept.

It has been found expedient to base unitary organization in mathematics upon the following general principles:[1]

1. Each unit of instruction is to have a central theme. The theme of the unit is a fundamental concept or process around which the instructional materials are organized.

2. Each unit of instruction is to have distinct, but associated unit elements. The elements of the unit are related processes and principles that will actually aid in the development, presentation, learning, and application of the theme.

3. Each unit of instruction is designed to result in definite and real learning products. The learning products become the aims of the unit.

4. Each unit of instruction is to be a significant part of the course. The course is thus constituted of related units. The course bears the same relation to the constituent units that the unit bears to its constituent elements.

5. The instructional materials of the units must be those exercises, problems, and applications that actually center about and focus upon the theme.

[1] Stone, C. A. and Georges, J. S., "General Principles of Unitary Organization," *School Science and Mathematics*, Vol. XXX, No. 8, pp. 901-906, November, 1930.

6. Each unit of instruction must be teachable at the school level where it is presented for instruction. The learning of the unit must actually attain its definite aims.

7. Each unit of instruction must provide means of identifying the presence of the attained learning products. The tests should be adequate to determine whether the learning products have or have not been attained.

The scope of this chapter will permit the discussion of only the first four principles, illustrating them in terms of the unit, *Direct Linear Variation.*

The theme of the unit, the concept of direct linear variation. The concept of function is of such fundamental significance in mathematics, and of such universal interest in non-mathematical fields of thought, that some phase of this rich concept should be made the basis of the organization of a unit of instruction and its entire study be made as a sequence of related units. The relationship $y = ax$ is presented here as a typical unit of that sequence, and specifically as defining the concept of direct linear variation.

The linear function $y = ax$ may be interpreted as a process in the study of variation and dependence of a specified nature. The study of this function involves phenomena and relationships that are assumed to contain but two variable factors, associated together by a relation whose nature is well defined and is expressible in mathematical symbolism. The variables y and x represent the two variable factors, and the parameter a represents the specific mode of variation between the two variable factors. The statement of the relation between y and ax as well as between x and $\frac{y}{a}$ gives the law of variation.

The relation $y = ax$ defines direct linear variation between the variables y and x, and it represents the simplest and the most satisfactory way of stating simple proportionality. The concept defines a definite and complete process by which certain types of variations can be studied. The concept thus presents definite learning products in terms of its various manifestations and applications.

The elements of the unit. The elements of the unit have been determined upon the basis of their logical connections with the theme of the unit, the concept of direct linear variation. They have been selected also because of their significance both in mathematics as basic concepts and processes, and in all scientific fields

as definite processes of quantitative thinking. The determination of their values in instruction has been based upon the following analyses: (1) books on the teaching of mathematics; (2) articles dealing with the concept of function; (3) selected textbooks in algebra and general mathematics. The six elements of the unit are:

1. E_1—Tabular representation of direct linear variation.
2. E_2—Graphic representation of direct linear variation.
3. E_3—Algebraic representation of direct linear variation.
4. E_4—The algebraic function $y = ax$.
5. E_5—The ratio form $\frac{y}{x} = a$.
6. E_6—The proportion form $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

The first three elements of the unit as presented here show how these processes, which are in general use in all branches of scientific and quantitative thinking, may be applied to the study of direct linear variation. The last three elements are specific algebraic forms representing direct linear variation.

Element E_1. The tabular method, which the pupil may have studied in previous work in mathematics, is presented here in its relation to direct linear variation. A table is considered a convenient way of showing him two related sequences of numbers. A number-pair defines two corresponding numbers. Our problem is to find the nature of the correspondence, that is, the relationship which holds throughout the table. The pupil can see that the tabular method, as presented in the unit, can be applied in the study of industrial, commercial, scientific, social, and educational problems which obey the law of direct linear variation. Furthermore, the method can be used to determine whether or not a given table of numerical facts displays this sort of variation.

The three methods of determining the relationship between the number-pairs of a table, applicable to the present unit, are: (1) the method of differences, where the differences of the corresponding number-pairs are compared; (2) the method of averages, where the averages of the number-pairs are compared; and (3) the method of ratios, where the ratios of the corresponding numbers of the table are compared. While all three methods are applicable to the relation $y = ax$, the first two are shown to be applicable to the general linear function $y = ax + b$, thus distinguishing between these two

types of relationships. Furthermore, the methods are used to determine when the relation expressed by a given table is nearly linear of the type $y = ax$.

The law of variation of a table, or the formula which states the relationship between the elements of a table, is determined by means of the ratio $\frac{y}{x}$, where y and x represent any corresponding elements of the table. If this ratio is the same for all the number-pairs, the law is stated in the form $\frac{y}{x} = a$. It is shown that the form $\frac{y}{x} = a$ is equivalent to the form $y = ax$. Direct linear variation is specifically defined in terms of the function $y = ax$.

Element E_2. In E_2 the graphic method of studying direct linear variation is considered. Pupils are shown that this method is to be used to represent related facts geometrically. It not only illustrates the relation between the two variables, but also enables comparisons as in a table. Moreover, the graph is shown to furnish additional facts which are not obvious in the table. It has the advantage of being thus more compact than a table.

Though the graphic method may have been studied previously, special attention is paid in the unit to the making of a graph to show the one-to-one correspondence between the number-pairs of the table and the points of the graph. Tabulation, selection of the reference lines, selection of suitable scales, plotting the points, and drawing the graph are the processes which are reviewed in their relation to direct linear variation. The terms *coördinate axes*, *ordinate*, and *abscissa* are used. Directed numbers are used in dividing the plane into four quadrants.

In interpreting the graph, special emphasis is placed upon the relation between the straight line graph and a table representing direct linear variation. The pupils are shown that any algebraic expression which can be written in the form $y = ax$ has a straight line for its graph, and that the graph intersects the coördinate axes at (0, 0).

The rate of change of the variable y with respect to the variable x is found by determining the ratio of the y-value, or the ordinate, to the x-value, or the abscissa, for any point on the graph. This ratio is seen to be the same for all the points on a straight line graph representing direct linear variation, just as it was true for all the number-pairs of the table. The rate of change is also in-

terpreted in terms of the tangent of the angle formed by the graph and the x-axis.

The constant of variation a is identified and associated with the rate of change, and with the ratio $\frac{y}{x}$. By drawing several graphs using the same coördinate axes and the same scales, it is shown that the rate of change depends only upon the particular value of the parameter a. Thus it is shown that specific values of the parameter a represent specific applications of the law of direct linear variation.

Element E_3. When the nature of any relation between two variable quantities is completely determined, the elation can be expressed algebraically. In Element E_3 derivations and interpretations of formulas representing direct linear variation are considered. In addition to the derivation of formulas from tables and graphs, which was considered in the Unit Elements E_1 and E_2, this part of the unit treats the translation of statements of variation into algebraic symbolism, and conversely, formulas are translated into statements of variation and dependence.

The statements of direct linear ariation may be of the following types: (1) specific relations explicitly stated, for example, the weight of water is 62.5 pounds per cubic foot; (2) implied relations, for example, a train travels at a uniform rate; (3) statements of dependence, for example, the perimeter of a square depends upon the length of the side; (4) statements of direct variation, for example, the weight of an object varies directly as the density; and (5) statements of proportionality, for example, the weight of copper in an alloy of copper and zinc is directly proportional to the weight of the zinc. In each case suitable symbols are selected for the variable and constant quantities, and the relationships are expressed algebraically. Each formula assumes one of the forms $y = ax$, $\frac{y}{x} = a$, or $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.

Throughout the Unit Element E_3 the formula is interpreted as a standard of exactness in the statement of functionality.

In the evaluation of the formula, products and quotients of integral and rational numbers, both absolute and directed, are presented.

Element E_4. Here the relation $y = ax$ is associated with the meaning of function, that is, the expression ax is a function of x, and y is a symbol for $f(x)$. It is shown that as x varies over its

range of values, the function ax, that is y, varies over its range of values in such manner that the relation $y = ax$ always holds. The function ax is compared with other algebraic functions, both linear and of higher degree, to emphasize its nature, and to enable the student to recognize this function out of a multitude of other algebraic functions.

In the form $y = ax$ the nature and the meaning of the symbols are studied: x is interpreted as the independent variable, y, the dependent variable, and a, an arbitrary constant. A true understanding of these ideas facilitates the transformation of a formula from one form into another, so that the independent variable of one form becomes the dependent variable of another form. Thus the idea of dependence is closely associated with that of functionality. If y is a function of x, the equation may be so transformed that x is expressed as a function of y. It is shown that this is always possible in direct linear variation.

The applications of the form $y = ax$ utilize the processes of evaluation and the solution of equations. Evaluation is the finding of specific functional values for assigned or determined particular values of the independent variable. The solution of the linear equation, on the other hand, is used to show that for given values of the dependent variable, the corresponding values of the independent variable may be readily determined.

The linear equation in one variable, namely $ax = b$, is shown to be but a special case of the linear function $y = ax$. This point of view needs further amplification. While traditionally the equation in one unknown is made the basic principle about which are built up and developed related algebraic manipulations and applications, the present point of view considers the general function in two variables as the basic principle and reduces the equation in one unknown to a specific application of the general law. The equation asks the question: What is the specific value of the independent variable x which corresponds to a specific value of the dependent variable y? This point of view has governed the selection of the instructional materials. Instead of the puzzle problems based upon the "popular sport of hunting the unknown," such as the classical number and age problems, practical applications and illustrations of direct linear variation from various sources are introduced in this connection which in their particularities yield equations in one unknown.

The method of sums and differences is resorted to in presenting equations of the form $a_1x \pm c_1 = \pm a_2x \pm c_2$, which are reducible to the form $ax = c$. One of the important functions of mathematical thought is the generalization of specific cases, more so in functionality than elsewhere; hence special provision is made to develop the habit of recognizing generalities of similar situations, and of interpreting like specific cases in terms of the general law covering them.

Element E_5. While in the development of the relation $y = ax$ the meaning and significance of the constant of variation a has been made evident in the previous elements of the unit, in E_5 the emphasis is placed on the ratio $\frac{y}{x}$. The variable ratio $\frac{y}{x}$ represents various particular manifestations of the law of direct linear variation. Thus the law of uniform motion, $d = rt$, may be considered as applying to many specific situations, each case being an appropriate interpretation of the law. For example, $\frac{d}{t} = 4$ may apply to the case of a man walking at the rate of 4 miles an hour; $\frac{d}{t} = 30$, to an automobile; $\frac{d}{t} = 50$, to a train; $\frac{d}{t} = 150$, to an aeroplane; and so on. Furthermore, since all these particular relations are expressible in the form $\frac{d}{t} = r$, the symbol r is a parameter which remains fixed for a given relation, but varies from relation to relation.

The values of a, which in reality represent the variable ratio $\frac{y}{x}$, are directly associated with the rate of change of y with respect to x, and with the slope or tangent of the graph.

The processes attendant upon the form $\frac{y}{x} = a$ are: (1) reduction of ratios or fractions; (2) evaluation; (3) solution of equations; and (4) transformation of formulas.

The transformation of formulas into the linear form $\frac{y}{x} = a$ requires the identification of the dependent and independent variables and the solution of the formula for the ratio $\frac{y}{x}$. Thus, $E{\cdot}S = R{\cdot}S'$

is put into the form $\frac{E}{R} = \frac{S'}{S}$ by solving the formula for the ratio $\frac{E}{R}$, the value of the constant a being represented by $\frac{S'}{S}$.

Element E_6. The form $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ which is treated in E_6 is the form in which the law of direct linear variation is generally known. It is in this form that the law is appropriately called the "law of simple proportionality." In many scientific problems, such as in physics, chemistry, and the social sciences, the proportional form is commonly used because it yields results simply and readily. In geometry, the concept of similarity is presented entirely by means of proportionality, and its use is carried to such an extent that often the law itself is not made evident to the student. Proportions are set up and solved without due regard to the law which governs similarity of geometric figures as a special case. In art, symmetry and harmony of line and form utilize the proportional form of the law. Thus it becomes perfectly clear that special emphasis is needed, first, to show the relation of the form to the law of direct linear variation, and, second, to enable its intelligent use as a process in the solution of problems. Without a clear understanding of the nature of the law, the use of proportion reduces to a juggling of numbers to obtain some kind of answer, often unreasonable.

Ample opportunity is provided in the unit to associate the proportional form $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ with: (1) the number-pairs of a table; (2) the coördinates of points on a graph; (3) statements of direct linear variation; and (4) the algebraic function $y = ax$. The assumption is made that the straight line law is completely determined from two specific situations. Calling the specific values of the variables y_1, x_1 in the first situation and y_2, x_2 in the second, the form $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ is readily obtained. Again, proceeding from the law $y = ax$, the proportional form is readily derived. For, since the law holds for all corresponding values of y and x, $y_1 = ax_1$ and $y_2 = ax_2$ or $\frac{y_1}{x_1} = \frac{y_2}{x_2}$, it is not necessary in many problems to solve for the constant of variation, the proportional form giving the desired result at once.

The applications of the form to specific problems are based upon the determination of proportions from tables, graphs, statements, and formulas. The resulting proportions yield the simple equations $\frac{y}{b} = \frac{c}{d}$, or $\frac{a}{x} = \frac{c}{d}$. Using sums and differences, we obtain such proportions as $\frac{(y_1 + y_2)}{(x_1 + x_2)} = \frac{y_1}{x_1}$.

The learning products of the unit. The learning products of the unit, the linear function $y = ax$, have been classified in terms of the elements of the unit into three main classes: the recognitions, the understandings, and the abilities. The recognitions are definite learnings which may or may not be associated with complete understanding of the concept. For example, the recognition of functionality in the algebraic expression may be associated with a complete understanding of the nature of the law if the expression is linear and has only two variables, but not if it has several variables. The understandings are interpreted in terms of reflective thinking about the concept. The rationalization of the concept, its significance, nature, and manifestations constitute the understandings of the unit if real learning has taken place. The abilities constitute the acquisition of skill in the necessary operations and the associated manipulations of the processes related to the fundamental concept.

Though in the presentation of the results of this study, and generally in a theoretical way, the learning products come first in the scheme of organization, their determination follows the selection of the theme and the identification of the related elements of the unit. It is very essential to distinguish between two procedures: the first which sets up the aims of the unit and then determines the necessary instructional materials, and the second which determines the instructional materials and then identifies the learning products, meanwhile eliminating those materials that do not yield actual learnings.

The recognitions of the unit are presented in Table I. They are classified into: (1) recognition of variation in general; (2) recognition of the types of variation; and (3) recognition of the linear variation $y = ax$. The check marks in the table indicate the elements of the unit which provide opportunities for the learnings associated with each item. For example, the recognition of variation in the statement, "one thing depends upon another thing," is

provided for throughout the unit, while the recognition of variation in the statement, "the greater the first factor, the greater the second factor," is treated only in the first three elements of the unit.

The understandings of the unit are presented in Table II and are classified as follows: (1) understanding the tabular method of studying variation; (2) the graphic method of studying variation; (3) the algebraic or symbolic method of studying variation; (4) understanding the linear form $y = ax$; (5) the linear form $\frac{y}{x} = a$; (6) the linear form $\frac{y_1}{x_1} = \frac{y_2}{x_2}$; and (7) understanding the application of the linear function to practical problems. The table presents the distribution of the various items of each class under the elements of the unit where opportunities are provided for the attainment of the understanding.

The resulting abilities are presented in Table III and are classified into: (1) abilities associated with the tabular method; (2) abilities associated with the graphic method; (3) abilities associated with the algebraic method; (4) ability to manipulate the various forms of the linear function $y = ax$; (5) ability to apply the formulas of the linear function $y = ax$; and (6) ability to solve the equations resulting from $y = ax$.

The unit a significant part of the course. The present unit is designed to be the first of a series of related units constituting a course in high school algebra. The various units have in common the interpretation and application of the concept of function. They differ in the manner of interpretation and application. Each unit presents a unique type of algebraic function.

A unit of instruction is an integral part of the course, and as such must make definite contributions toward the realization of the instructional aims of the course. The organization of educational courses by units might be thought of as parallel to the structure in units of biological organisms. The units are to the course what the various organs are to the organism. The evolutionary processes of nature do not first set up an organism and then apportion to it a set of organs to perform the essential functions. Hence, is it not unnatural to set up a course of instruction beforehand and then determine units for it which are to be its integral parts, as is usually done in the construction of courses? The attempts to take a traditional course of instruction, which was originally organized

TABLE I: THE LEARNING PRODUCTS AS RECOGNITIONS

Learning Product	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
I. *Recognition of variation.*						
A. In statements:						
1. ...depends upon...	√	√	√	√	√	√
2. ...varies as...	√	√	√	√	√	√
3. ...varies directly as...	√	√	√	√	√	√
4. ...is proportional to...	√	√	√			√
5. ...is directly proportional to...	√	√	√	√	√	√
6. ...is a function of...	√	√	√	√	√	√
7. The ratio is constant	√	√	√		√	√
8. ...increases as ... increases.	√	√	√	√	√	
9. ...decreases as ... decreases.	√	√	√	√	√	
10. The greater the ... the greater the ...	√	√	√			
11. The less the ... the less the ...	√	√	√			
B. In tables:						
1. ...varies directly as...	√	√	√	√	√	√
2. ...is a function of...	√	√	√	√	√	√
3. The ratio is constant.	√	√	√		√	√
4. ...is directly proportional to...	√	√	√			√
5. ...increases as ... increases.	√	√	√			
6. ...decreases as ... decreases.	√	√	√			
C. In graphs:						
1. ...varies directly as...		√	√	√	√	√
2. ...is a function of...		√	√	√	√	√
3. The ratio is constant.		√	√	√	√	√
4. ...is directly proportional to...		√	√	√	√	√
5. ...increases as ... increases.		√	√			
6. ...decreases as ... decreases.		√	√			
D. In formulas:						
1. ...varies directly as...	√	√	√	√	√	√
2. ...is a function of...	√	√	√	√	√	√
3. The ratio is constant.	√	√	√		√	√
4. ...is directly proportional to...	√	√	√			√
5. ...increases as ... increases.	√	√	√	√		
6. ...decreases as ... decreases.	√	√	√	√		
II. *Recognition of the types of variation.*						
A. Multiple-factor types in statements.				√	√	√
B. Two-factor types:						
1. In statements	√	√	√	√	√	√
2. In tables	√	√	√	√	√	√
3. In graphs		√	√	√	√	√
4. In formulas	√	√	√	√	√	√

TABLE I (*Continued*)

Learning Product	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
C. Direct linear variation:						
1. In statements	✓	✓	✓	✓	✓	✓
2. In tables	✓	✓	✓	✓	✓	✓
3. In graphs		✓	✓	✓	✓	✓
4. In formulas	✓	✓	✓	✓	✓	✓
III. *Recognition of the linear variation $y = ax$.*						
A. As a general law	✓	✓	✓	✓	✓	✓
B. As a special type	✓	✓	✓	✓	✓	✓

TABLE II. The Learning Products as Understandings

Learning Product	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
I. *Understanding the tabular method of studying variation.*						
A. Dependence of the elements.						
1. Variable factors of a table	✓	✓	✓	✓	✓	✓
2. Number-pairs	✓	✓	✓	✓	✓	✓
3. Relation of correspondence between number-pairs	✓	✓	✓		✓	✓
B. Methods of comparing the elements.						
1. Ratios of number-pairs	✓	✓	✓		✓	✓
2. Ratios of differences	✓	✓	✓			
3. Ratios of averages	✓					
C. Relation between a table and its graph.						
1. Number-pair corresponding to coordinates of a point	✓	✓	✓	✓	✓	✓
2. Ratio corresponding to the tangent		✓	✓			
D. The law of variation of a table.						
1. $y = ax + b$		✓				
2. $y = ax$	✓	✓	✓			
II. *Understanding the graphic method of studying variation.*						
A. The making of a graph.						
1. The coördinate axes		✓	✓	✓	✓	✓
2. The units		✓	✓	✓	✓	✓
3. The ordinate, the abscissa		✓	✓	✓	✓	✓
4. The correspondence between points and number-pairs		✓	✓	✓	✓	✓

TABLE II (*Continued*)

Learning Product	Unit Element E_1	E_2	E_3	E_4	E_5	E_6
B. Interpretation of a straight line graph.						
1. The form of the graph..........		√	√	√	√	√
2. The intersection with x-axis, y-axis		√	√	√		
3. The rate of increase............		√	√	√		
4. The tangent		√	√			
5. The formula, the law of variation.		√	√	√	√	√
III. *Understanding the algebraic or symbolic method of studying variation.*						
A. Algebraic statement of "depends upon"			√	√	√	√
B. Algebraic statement of "is a function of"			√	√		
C. Algebraic statement of "varies directly as"	√	√	√	√	√	√
D. Algebraic statement of "is directly proportional to"	√	√	√	√	√	√
E. Algebraic statement of "their ratio is constant"	√		√		√	
F. Algebraic expression of rules.......			√	√	√	
G. Formulas of tables of variation.....	√		√			
H. Formulas of graphs...............		√	√			√
I. Formulas as laws of variation.......	√	√	√	√	√	√
IV. *Understanding the form* $y = ax$.						
A. y as a function of x..............			√	√		
B. y varying directly as x...........	√	√	√	√	√	√
C. y as directly proportional to x......	√	√	√	√	√	√
D. Variables and constants.						
1. Independent variable x.........	√	√	√	√	√	
2. Dependent variable y..........	√	√	√	√	√	
3. Constant of variation a.........	√	√	√	√	√	
E. The transformation of $y = ax$ into $x = \frac{y}{a}$				√	√	√
F. The graph of $y = ax$...........		√	√	√		√
G. The equation $ax = b$.						
1. The special case of $y = ax$...				√		
2. The solution and the root.......				√		
V. *Understanding the form* $\frac{y}{x} = a$.						
A. Derivation of $\frac{y}{x} = a$ from:						

TABLE II (*Continued*)

Learning Product	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
1. Statement, "the ratio is constant."	√	√	√		√	
2. Table	√				√	
3. Graph		√			√	
B. The ratio $\frac{y}{x}$ a variable					√	
C. The ratio $\frac{y}{x}$ a constant	√	√	√	√	√	
D. Reduction of ratio $\frac{y}{x}$					√	
E. Transformation of $y = ax$ into $\frac{y}{x} = a$			√		√	
F. Transformation of $y = ax$ into $\frac{x}{y} = \frac{1}{a}$					√	
G. Solution of formulas for $\frac{y}{x}$					√	
H. Solution of formulas for a					√	
I. Solution of the equation $\frac{y}{a} = \pm b$					√	
J. Solution of the equation $\frac{a}{x} = \pm b$					√	
VI. *Understanding the form* $\frac{y_1}{x_1} = \frac{y_2}{x_2}$.						
A. Derivation of $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ from:						
1. Statement, "is directly proportional to"			√			√
2. Table	√					√
3. Graph		√				√
4. $y = ax$			√	√		√
5. $\frac{y}{x} = a$					√	√
B. The proportion $\frac{y_1}{x_1} = \frac{y_2}{x_2}$			√			√
C. Solution of the equation $\frac{a}{x} = \pm\frac{b}{c}$						√
D. Solution of the equation $\frac{y}{a} = \pm\frac{b}{c}$						√

TABLE II (*Continued*)

Learning Product	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
VII. *Understanding the application of linear function to practical problems.*						
A. Special cases of linear function......				√	√	√
B. The characteristic formula.........				√	√	√
C. Relations explicitly stated..........				√	√	√
D. Relations implied				√	√	√

TABLE III: The Learning Products as Special Abilities

Learning Product	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
I. *Abilities associated with the tabular method.*						
A. Ability to tabulate.						
1. Statements of variation	√	√	√	√	√	√
2. Graphic relations		√				
3. Algebraic relations			√	√	√	
B. Ability to interpret tables.						
1. Finding ratio of corresponding number-pairs	√	√	√		√	√
2. Finding ratio of corresponding differences of number-pairs........	√	√	√			
3. Finding ratio of corresponding averages of number-pairs..........	√					
4. Stating the law of variation......	√	√	√			
II. *Abilities associated with the graphic method.*						
A. Ability to make a graph.						
1. Selecting coördinate axes........		√	√	√		√
2. Selecting units		√	√	√		√
3. Plotting the points.............		√	√	√		√
4. Drawing the graph		√	√	√		√
B. Ability to interpret a graph.						
1. Determining ordinate for given abscissa		√	√	√		√
2. Determining abscissa for given ordinate		√	√	√		√
3. Determining coördinate of given points on the graph.............		√	√	√		√
4. Determining ratio of coördinates.		√	√	√		

Learning Product	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
5. Determining ratio of corresponding differences of coördinates		√	√	√		
6. Determining rate of increase.....		√	√	√		√
7. Determining the tangent		√				
8. Determining the law of variation.		√	√	√		√
III. *Abilities associated with the algebraic method.*						
A. Ability to express relations algebraically.						
1. Selecting letters for variables....	√	√	√	√	√	√
2. Representing constants	√	√	√	√	√	
3. Expressing statements:						
a. Depends upon...			√	√		
b. Varies directly as...	√	√	√	√	√	√
c. Is directly proportional to... ...	√	√	√		√	√
d. The ratio is constant	√		√		√	
e. Is a function of...			√	√		
4. Determining formulas from tables.	√		√			
5. Determining formulas from graphs.		√	√			
B. Ability to interpret formulas.						
1. Stating the laws of variation in words			√	√	√	√
2. Determining the independent variable			√	√		
3. Determining the dependent variable			√	√	√	
4. Determining the constant........	√	√	√	√	√	
IV. *Ability to manipulate the various forms of the linear function $y = ax$.*						
A. Ability to evaluate formulas.						
1. For the dependent variable......	√	√	√	√		
2. For the independent variable.....	√	√	√	√		
3. For the ratio $\frac{y}{x}$	√	√	√		√	√
4. For the ratio $\frac{x}{y}$			√		√	
5. For the constant a	√	√	√	√	√	√
B. Ability to transform formulas.						
1. Into the form $y = ax$..........			√	√	√	
2. Into the form $\frac{y}{x} = a$...........			√	√	√	
3. Into the form $\frac{y_1}{x_1} = \frac{y_2}{x_2}$..........			√			√

TABLE III (*Continued*)

LEARNING PRODUCT	UNIT ELEMENT					
	E_1	E_2	E_3	E_4	E_5	E_6
C. Ability to solve formulas.						
1. For the dependent variable			√	√	√	
2. For the independent variable.....			√	√	√	
3. For the constant a			√	√	√	
4. For the ratio $\frac{y}{x}$...............			√	√	√	√
V. *Ability to apply the formulas of the linear function $y = ax$.*						
A. Relations explicitly stated..........				√	√	√
B. Special cases				√	√	√
C. Relations implied				√	√	√
VI. *Ability to solve the equations resulting from $y = ax$.*						
A. $ax = \pm c$	√	√	√	√		
B. $\frac{ax}{b} = \pm c$				√	√	
C. $ax = \pm \frac{c}{d}$	√	√	√	√		
D. $\frac{ax}{b} = \pm \frac{c}{d}$				√	√	√
E. $\frac{a}{bx} = \pm c$					√	√
F. $\frac{a}{bx} = \pm \frac{c}{d}$....................					√	√
G. $a_1x + a_2x = c$				√		
H. $a_1x = a_2x \pm c$				√		
I. $a_1x \pm c_1 = a_2x \pm c_2$				√		
J. $\frac{a_1x}{b_1} \pm \frac{c_1}{d_1} = \frac{a_2x}{b_2} \pm \frac{c_2}{d_2}$............					√	√

upon an entirely different plan—the chapter or topic plan—and create units for it will prove unsuccessful, for it is unnatural. The course divisions, thus determined, may be labeled units by the text-book writer or the course maker, but nevertheless they are the original chapters or topics under a new name. Changing the name of the parts, or painting them different colors, may alter the appearance of the structure but not the structure itself.

The concept plan which is the basis of the present organization brings together those methods, principles, forms, and applications

that can be homogeneously molded together to present the concept in its entirety. Such a body of instructional materials constitutes a unit made up of definite elements. A sequence of these related concepts, resulting in closely associated units, forms the course.

To determine whether a unit is or is not a significant part of the course, a very simple test will suffice. Omit the particular unit from the course and observe the effect on the course. A course constructed on the chapter or topic plan will not materially suffer by such an omission, while a course consisting of a sequence of units is entirely wrecked, for the sequence is destroyed when any one of its elements—a unit—is removed. This is obvious from the fact that each element of a sequence is based upon and is a consequence of the preceding element.

The linear function $y = ax$ as a significant part of algebra. The full realization of the significance of a unit lies in understanding its relation to the unique functions of the subject of which it is a part. In other words, the aims of a subject are only attainable through the contributions of the separate units. Whether the unit, direct linear function, is or is not a significant part of algebra depends upon what educational as well as what mathematical values we may assign to the subject, and whether the unit contributes significantly to the realization of these aims.

While space will not permit a full discussion of the problem of the determination of the aims of mathematical instruction in general and of algebra in particular, nevertheless, there are certain general and specific aims, values, and purposes commonly attributed to the subject which might be used as a basis of comparison and evaluation. For example, we may use the list of abilities, understandings, and appreciations representing the crystallized opinions of mathematicians and teachers of mathematics, as reported by the National Committee on Mathematical Requirements.[2] Their list follows:

A. Practical aims, i.e., of immediate or direct usefulness in life.
1. Developing the ability to apply the fundamental processes of arithmetic.
2. Understanding the language of algebra.
3. Developing the ability to understand and use algebraic technique.
4. Developing the ability to understand and use graphs.
5. Developing familiarity with geometric forms.
6. Acquiring the ability to understand and use quantitative ideas.

[2] *The Reorganization of Mathematics in Secondary Education*, p. 10, a report issued under the auspices of The Mathematical Association of America, Inc., 1923.

B. Disciplinary aims, i.e., related to mental training.
   1. Acquiring ideas and concepts used in quantitative thinking.
   2. Developing the ability to think in quantitative terms.
   3. Acquiring mental habits and attitudes useful in functional thinking.
   4. Acquiring the concept of dependence and relationship.

C. Cultural aims.
   1. Appreciating beauty in the geometric forms of nature, art, and industry.
   2. Acquiring ideals of perfection.
   3. Appreciating the power of mathematics.

Table IV presents a clear picture of the treatment in the various elements of the present unit of the general aims of mathematical instruction recommended by the National Committee. It is to be noted that A5 and C1 are the only two items not considered in the unit and they are purely geometric aims.

TABLE IV

AIMS OF THE NATIONAL COMMITTEE CONSIDERED IN THE PRESENT UNIT

Committee List	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
A1	√	√	√	√	√	√
A2	√	√	√	√	√	√
A3			√	√	√	√
A4		√	√	√	√	√
A5						
A6	√	√	√	√	√	√
B1	√	√	√	√	√	√
B2	√	√	√	√	√	√
B3	√	√	√	√	√	√
B4	√	√	√	√	√	√
C1						
C2	√	√	√			
C3	√	√	√			

A list of the concepts, processes, and principles of algebra which has been compiled by the writer from the literature available [3] is presented in Table V in order to show how they are treated in the present unit.

As Whitehead has well said, "Algebra is the intellectual instrument which has been created for rendering clear the quantitative

[3] Georges, J. S., "Mathematics in the Scheme of General Education," *School Science and Mathematics*, Vol. XXXII, No. 1, pp. 57-64, January, 1932.

TABLE V

ALGEBRAIC CONCEPTS, PROCESSES, AND PRINCIPLES CONSIDERED IN THE PRESENT UNIT

Concept or Process	Unit Element					
	E_1	E_2	E_3	E_4	E_5	E_6
1. Algebraic representation			√	√	√	√
2. Application of formulas			√	√	√	√
3. Accuracy	√	√	√	√	√	√
4. Approximation	√	√				
5. Comparison of data, method of differences	√	√		√	√	
6. Comparison of data, method of averages	√					
7. Comparison of data, method of ratios	√	√	√	√	√	√
8. Combination of terms				√	√	√
9. Correspondence	√	√	√	√	√	√
10. Directed number	√	√	√	√	√	√
11. Direct variation	√	√	√	√	√	√
12. Errors	√					
13. Evaluation	√	√	√	√	√	√
14. Exponential representation						
15. Factoring					√	
16. Fundamental operations	√	√	√	√	√	√
17. Functionality	√	√	√	√	√	√
18. Generalization			√	√	√	√
19. Graphic representation		√	√	√	√	√
20. Inverse variation						
21. Joint variation				√	√	√
22. Laws of variation	√	√	√	√	√	√
23. Limits						
24. Linear function	√	√	√	√	√	√
25. Logarithmic computation						
26. Maxima and minima						
27. Number	√	√	√	√	√	√
28. Probability						
29. Progressions						
30. Proportional representation	√	√				√
31. Quadratic function						
32. Range of variation		√				
33. Rate of variation	√	√	√	√	√	
34. Ratio	√	√	√	√	√	√
35. Special cases	√	√	√	√	√	√
36. Systems of equations						
37. Solution of equations			√	√	√	√
38. Tabular representation	√	√	√	√	√	√
39. Transformation					√	√
40. Variables and constants	√	√	√	√	√	√

aspects of the world." Function is the method of algebra in studying and stating the quantitative relationships of the concrete world of form and the abstract world of ideas. A unit which considers an aspect of the fundamental concept of function is indeed a very significant part of algebra. Tables IV and V justify this claim for the present unit of instruction.